

Which Holonomy Signatures Are Realizable?

A Complete Answer for Closed Surfaces

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Abstract

A seamless parametrization of a closed oriented surface carries a discrete invariant, its *holonomy signature*: the cone angles, all multiples of $\pi/2$, together with the rotational holonomy $\rho: H_1(M \setminus C) \rightarrow \mathbb{Z}_4$ of the induced cross field. This is the datum a quadrangulation prescribes. It does not determine the parametrization, since structures sharing a signature come in positive-dimensional families, but it does decide whether any parametrization exists at all. Shen, Zhu, Capouellez, Panozzo, Campen and Zorin asked which signatures occur and gave a sufficient condition of gcd type. Which signatures are realizable has remained open, and the reason is visible in the shape of what was known: the sufficient conditions are combinatorial, while the one obstruction, a torus with cones of angles $3\pi/2$ and $5\pi/2$, is conformal.

We answer the question. A Reduction Lemma shows that the mapping class group acts on signatures with fixed cone angles with orbits classified by the subgroup $\text{im } \rho \leq \mathbb{Z}_4$ alone, so at most three cases survive per angle multiset instead of 4^{2g} . A dictionary then identifies seamless parametrizations with meromorphic 4-differentials, under which $\text{im } \rho$ measures primitivity, and realizability becomes non-emptiness of a stratum of primitive k -differentials with $k = 4/d$ and $\text{im } \rho = \langle d \rangle$. Unwinding this against the known classification of such strata leaves exactly five exceptional families; every other admissible signature is realizable, in every genus. Two of the five appear to be new, and both live in genus two. Four of the five lie outside the gcd condition, and §5 settles the whole region that condition leaves open.

The non-emptiness half is also made constructive. We exhibit an explicit one-vertex square-tiled surface in every genus, together with a local surgery that cuts a saddle connection from a cone to itself and glues in a single square, splitting one cone into two of prescribed angles while leaving the genus, the other cones and $\text{im } \rho$ alone. Iterating produces quad meshes of the smallest size the corner count permits. We do not prove that the iteration always terminates, so the classification cites its non-emptiness input rather than reproving it.

Two extensions follow. On surfaces with boundary, which is the setting of parametrization aligned to feature curves, the Reduction Lemma holds verbatim once the boundary turnings are allowed into the relevant subgroup, and a feature component with an odd number of odd-angle corners makes the holonomy irrelevant. At a fixed conformal structure the holonomy is not free but computed, by an Abel–Jacobi condition. That is what explains why cones of odd topological valence lie beyond methods built on quadratic differentials.

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1 Introduction

1.1 The problem

Let M be a closed oriented surface of genus g and let $C = \{c_1, \dots, c_n\} \subset M$ be a finite set of marked points. A *seamless parametrization* of (M, C) is a locally injective atlas on $M \setminus C$ with values in $\mathbb{R}^2$ whose transition maps lie in $\mathbb{Z}_4 \ltimes \mathbb{R}^2$, that is, are of the form $z \mapsto i^k z + t$, and whose cone angles at the points of C are multiples of $\pi/2$. Such an atlas is what a quadrangulation or a quad-dominant texture chart is built on. The transitions fix the two grid *directions* up to a quarter turn, so the cross field of axis directions pulls back consistently. The integer *lattice* does not, since an arbitrary translation t moves it, and aligning all charts to one $\mathbb{Z}^2$ is the separate quantization step of §6.3.

The topological data of such an atlas is its *holonomy signature*: the cone angles $\theta_i = (m_i + 4)\pi/2$, recorded by the integers $m_i > -4$, together with the rotational holonomy

$$\rho: H_1(M \setminus C; \mathbb{Z}) \longrightarrow \mathbb{Z}_4, \quad \rho(\gamma_i) \equiv m_i \pmod{4},$$

where γ_i is a small loop around c_i . Equivalently, ρ is the holonomy of the cross field of the parametrization. Shen, Zhu, Capouellez, Panozzo, Campen and Zorin [14] asked:

For which holonomy signatures do seamless parametrizations with the corresponding topology exist?

They proved a sufficient condition of gcd type. In their normalization a cone of angle $(m+4)\pi/2$ has index $I = -m/4$, and the condition [14, Prop. 2] is $\gcd(I_1, \dots, I_n) = \frac{1}{4}$, that is

$$\gcd_{\mathbb{Z}}(m_1, \dots, m_n) = 1.$$

Under it every signature is realizable, in every genus, with a single exception: the torus with exactly two cones, of angles $3\pi/2$ and $5\pi/2$. The condition holds as soon as one cone has angle $3\pi/2$ or $5\pi/2$, since then some $|m_i| = 1$. The authors note that it is not necessary, and single out “indices restricted to multiples of $\frac{1}{2}$ ”, our even case, as a realistic scenario it does not cover. The exception itself is a theorem of Izvestiev, Kusner, Rote, Springborn and Sullivan [7], who proved the equivalent combinatorial statement that the torus admits no 3,5-quadrangulation. The signature formulation and the constructions built on it have since been developed further, for arbitrary cone configurations in [1], in Penner coordinates in [2], and for surfaces with marked feature curves in [3], but always on the sufficient side.

The gap has a definite shape. The known sufficient conditions are combinatorial, resting on a rerouting argument, while the known obstruction is conformal and algebraic. This paper closes the gap by working throughout in the language where the obstruction naturally lives.

1.2 Results

Write $D = \langle m_1, \dots, m_n \rangle \leq \mathbb{Z}_4$ for the subgroup generated by the cone data and d for the generator of $\text{im } \rho$ in $\{1, 2, 4\}$, so that $\text{im } \rho = d\mathbb{Z}_4$. Since $\rho(\gamma_i) = m_i$, every m_i lies in $\text{im } \rho$, so $D \subseteq d\mathbb{Z}_4$; writing $D = d_D\mathbb{Z}_4$ this says $d \mid d_D$.

(a) The holonomy data collapses. Theorem 3.4, the Reduction Lemma, states that for $g \geq 1$ two signatures with the same genus and the same cone angles lie in the same $\text{MCG}(M, C)$ -orbit, after relabelling the marked points to match the orders, if and only if they have the same $\text{im } \rho$. Figure 1 shows what the three possibilities mean for the cross field. Since realizability is a mapping class group invariant, the problem depends on at most three cases per angle multiset instead of 4^{2g} . Corollary 3.5 is the special case that makes contact with the literature: if some m_i is odd, in particular if some cone has angle $3\pi/2$ or $5\pi/2$, then $D = \mathbb{Z}_4$, a single orbit remains, and ρ is irrelevant. This explains the shape of the gcd condition of [14] from first principles, and it also shows that condition is more than is needed: coprimality of the m_i is what their rerouting requires, one odd m_i is what realizability requires.

(b) Realizability is non-emptiness of a stratum. Theorem 3.6: a signature is realizable if and only if the stratum of *primitive* $(4/d)$ -differentials with orders m_i/d on a genus- g surface is non-empty. The proof runs through the dictionary of §2.4, in which a seamless parametrization is a flat cone metric with holonomy in $\mathbb{Z}_4$, hence a meromorphic section q of K^4 with a zero or pole of order m_i at c_i , and in which $\text{im } \rho$ measures exactly how far q is from being a power.

(c) The classification. Unwinding Theorem 3.6 with the known non-emptiness classifications for abelian and quadratic differentials gives Theorem 3.10: a signature fails to be realizable precisely in the five families of Table 1, in every genus. For $g \geq 1$ the non-emptiness input is a

single cited theorem, (S7), which covers all three values of k at once and whose genus-one half is reproved here independently; genus 0 is separate, and is Proposition 3.7. Two of the five are in genus 1 and are proved here from an Abel–Jacobi criterion (Proposition 3.8); one of them is the exception of [14, 7], recovered in one line. The other two live in genus 2 and appear to be new in this setting: they are the images under the dictionary of the empty quadratic strata $Q(4)$ and $Q(1, 3)$ of [13]. In the language of cross fields, a genus-two surface carrying a cross field whose holonomy along every homology loop is a multiple of π cannot be quadrangulated if its singularities are a single cone of angle 6π , or a pair of cones of angles 3π and 5π .

(d) An explicit construction. §4 makes the non-emptiness half constructive. We exhibit an explicit square-tiled surface with one cone in each genus (Lemma 4.3) and a local surgery (Lemma 4.1) that cuts a saddle connection from a cone to itself and glues in one unit square, splitting a cone of angle $w\pi/2$ into two of angles $(\gamma + 2)\pi/2$ and $(w - \gamma + 2)\pi/2$ while leaving the genus, every other cone and $\text{im } \rho$ untouched. Iterating with backtracking (Algorithm 4.5) produces a quad mesh with $2g - 2 + n$ squares, the minimum the corner count allows, and each output is a witness whose genus, cone orders and $\text{im } \rho$ can be read back off the gluing alone (Theorem 4.6). We do not prove that the iteration always terminates, so this is not by itself a theorem for all genera.

(e) Feature curves. §6.1 redoes the theory for compact surfaces with boundary whose boundary must develop to axis-parallel segments, the setting a feature network cuts a surface into. Gauss–Bonnet gains a corner term and forces a parity constraint (Lemma 6.1); the space of signatures is again $\mathbb{Z}_4^{2g}$ (Lemma 6.2); and the Reduction Lemma holds verbatim with D enlarged by the boundary turnings (Theorem 6.3). Consequently a boundary component of odd turning, equivalently one carrying an odd number of corners of odd angle, forces $D = \mathbb{Z}_4$, so the holonomy along homology loops cannot obstruct anything (Corollary 6.4).

(f) Relation to the fixed-structure criterion. §6.2 shows that at a fixed Riemann surface with fixed cone positions the holonomy is not free at all but is *computed* by an Abel–Jacobi condition (Proposition 6.7), so the criterion of that line of work [11, 12] is the pointwise condition and Theorem 3.6 asks whether the locus it cuts out is non-empty. The same statement explains why cones of odd topological valence are inaccessible to methods built on quadratic differentials (Corollary 6.9).

1.3 What this adds to the sufficient condition

Since [14] is both the source of the question and the only prior general answer, it is worth saying precisely what changes. Their Proposition 2 is a one-way statement: if $\gcd_{\mathbb{Z}}(m_1, \dots, m_n) = 1$ then the signature is realizable, with the single torus exception. Outside that hypothesis it says nothing, and as the paper itself notes the condition “is not necessary”. Theorem 3.10 fills the empty cell:

	$\gcd_{\mathbb{Z}}(m_i) = 1$	$\gcd_{\mathbb{Z}}(m_i) \neq 1$
[14, Prop. 2]	realizable; 1 exception	no statement
here	realizable; the same 1 exception	realizable; 4 exceptions

Three things are behind that, and one thing is not improved.

(i) Sufficient becomes necessary and sufficient. A combinatorial construction can certify that a signature is realizable; it cannot certify that one is not. Theorem 3.6 converts the question

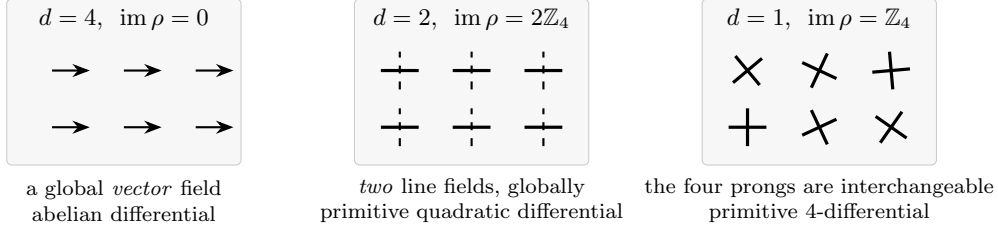


Figure 1: The three possible values of d , the generator of $\text{im } \rho$, and what each means for the cross field. When $\text{im } \rho = 0$ the cross lifts to a global vector field. When $\text{im } \rho = 2\mathbb{Z}_4$ it does not, but it still splits globally into two distinguishable line fields, drawn solid and dashed. When $\text{im } \rho = \mathbb{Z}_4$ transport around some loop cyclically permutes the four prongs and no such splitting exists. By the Reduction Lemma these three cases are all that matter: for fixed cone angles, $\text{im } \rho$ alone decides the mapping class group orbit, so realizability is a question about at most three signatures per angle multiset rather than 4^{2g} . Families 4 and 5 of Table 1 live in the middle column.

into non-emptiness of a stratum of primitive k -differentials, where the empty strata have been classified, and a complete list follows: exactly five families fail, in every genus (Table 1). One of the five is the torus exception of [14, 7]; the other four are the content of the right-hand column above. Restricted to $\gcd_{\mathbb{Z}}(m_i) = 1$ the classification returns their statement, so nothing is contradicted, though not independently of them, since our positive half cites (S7) where theirs is constructive.

(ii) *Where the two overlap, less is needed.* Their rerouting moves a loop's turning number by $I_i = -m_i/4$ for each cone it is rerouted around, so the reachable adjustments are the integer combinations of the I_i , and hitting an arbitrary target in $\frac{1}{4}\mathbb{Z}$ forces $\gcd_{\mathbb{Z}}(m_i) = 1$. It has to be an arbitrary target because the construction of [1] that they build on emits one fixed holonomy pattern along the cut graph, which the signature must be made to match exactly.

The corresponding hypothesis here is weaker, and the reason is the difference between a total and a residue. A turning number in $\frac{1}{4}\mathbb{Z}$ is a count of quarter turns, in which one full turn and three full turns are different numbers; realizability, by Lemma 2.4, depends only on the rotation class in $\mathbb{Z}_4$, where they are the same. Point pushing changes that class by m_i , so Corollary 3.5 needs only $\langle m_1, \dots, m_n \rangle = \mathbb{Z}_4$, one odd m_i , with no coprimality. The gap is not hypothetical: at $g = 4$ the cone angles $7\pi/2$ and $25\pi/2$ give $m = (3, 21)$, so $\gcd_{\mathbb{Z}} = 3$ and Proposition 2 does not apply, while 3 is odd, so a single mapping class group orbit remains, ρ is irrelevant, and the signature is realizable on eight squares, the minimum.

(iii) *The region left open is decided.* The condition $\gcd_{\mathbb{Z}}(m_i) \neq 1$ is larger than it looks. It contains every signature whose cone angles are all multiples of π , the case [14] singles out as realistic and does not cover; every signature with a single cone, since there $\gcd_{\mathbb{Z}}(m) = |m_1| = 4(2g - 2)$, which is never 1; and, by (ii), signatures with an odd cone angle as well. Corollary 5.1 settles all of it at once: realizable except for four signatures, all in genus 1 or 2. §5 works through the cases. The one to keep in mind is a genus-two surface with a single cone of angle 6π , where the answer is yes for $\text{im } \rho = \mathbb{Z}_4$, no for $2\mathbb{Z}_4$, and yes for 0. Same surface, same singularity, and the holonomy alone decides. That is also why the reduction from 4^{2g} signatures to three, item (a) above, is doing real work rather than tidying.

What is not improved. [14] is an algorithm that runs on meshes, producing parametrizations

whose distortion is then optimized; the present paper decides existence and, in §4, builds minimal witnesses on a finite range without a termination proof. On their own region their result is constructive and self-contained where ours cites (S7). The two are complementary in the ordinary way: they answer how to build one, we answer whether there is one to build.

2 Seamless parametrizations and 4-differentials

2.1 Conventions

M is a closed oriented surface of genus g , and $C = \{c_1, \dots, c_n\} \subset M$ a set of distinct marked points, *labelled*: the orders $m = (m_1, \dots, m_n)$ are an ordered vector attached to $c_1, \dots, c_n$. Accordingly $\text{MCG}(M, C)$ denotes the group of isotopy classes of orientation-preserving homeomorphisms of M fixing C *pointwise*; it does not permute the marked points. Relabelling by a permutation π of equal-order points is a separate operation, used only where stated, and it identifies the signature (m, ρ) with $(m \circ \pi, \rho \circ \pi)$. Statements phrased in terms of the multiset of cone angles are to be read modulo that identification. We write $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ additively and identify it with the group μ_4 of quarter turns, $1 \in \mathbb{Z}_4$ being the rotation by $\pi/2$. Subgroups of $\mathbb{Z}_4$ are written $d\mathbb{Z}_4$ with $d \in \{1, 2, 4\}$, so $4\mathbb{Z}_4 = 0$.

Definition 2.1. A *holonomy signature* on (M, C) is a pair $s = (m, \rho)$ where

- $m = (m_1, \dots, m_n)$ are integers $m_i > -4$, the *orders*, encoding cone angles $\theta_i = (m_i + 4)\pi/2$ and subject to Gauss–Bonnet $\sum_i m_i = 4(2g - 2)$;
- $\rho: H_1(M \setminus C; \mathbb{Z}) \rightarrow \mathbb{Z}_4$ is a homomorphism with $\rho(\gamma_i) = m_i \bmod 4$.

An order $m_i = 0$ is a *marked point*: the metric is smooth there.

Since $H_1(M \setminus C; \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus (\bigoplus_i \mathbb{Z}\gamma_i)/(\sum_i \gamma_i)$ and Gauss–Bonnet gives $\sum_i m_i \equiv 0 \pmod{4}$, such a ρ exists, and the set

$$A(m) = \{\rho : \rho(\gamma_i) = m_i \forall i\}$$

is a coset of $H^1(M; \mathbb{Z}_4)$ inside $H^1(M \setminus C; \mathbb{Z}_4)$, the inclusion $H^1(M; \mathbb{Z}_4) \hookrightarrow H^1(M \setminus C; \mathbb{Z}_4)$ being injective. Signatures are thus *characters*, and the space they form is cohomological. Fixing a symplectic basis $a_1, b_1, \dots, a_g, b_g$ of $H_1(M; \mathbb{Z})$ identifies $H^1(M; \mathbb{Z}_4)$ with $\mathbb{Z}_4^{2g}$, and Poincaré duality on the closed oriented surface M identifies it with $H_1(M; \mathbb{Z}_4)$, carrying the cup product to the intersection form. We use that identification silently below: the symplectic group acts on $H^1(M; \mathbb{Z}_4) \cong \mathbb{Z}_4^{2g}$, while curves, meaning the loops γ_i , the a_j, b_j , and the arcs along which cones are pushed, live in H_1 .

A seamless parametrization is *realized on a mesh* if it is piecewise linear with respect to some refinement of a given triangulation of M whose vertex set contains C . A signature is *realizable* if some seamless parametrization has it. We also use the following invariants of a signature:

$$D = \langle m_1 \bmod 4, \dots, m_n \bmod 4 \rangle \leq \mathbb{Z}_4, \quad \text{im } \rho = d\mathbb{Z}_4 \quad (d \in \{1, 2, 4\}), \quad D \subseteq \text{im } \rho.$$

2.2 Quad meshes

Remark 2.2 (square-tiled surfaces). A closed quad mesh built from N unit squares is a seamless parametrization: each square is a chart, the transitions are rotations by multiples of $\pi/2$ and translations, and a vertex of valence v is a cone of angle $v\pi/2$, that is of order $m = v - 4$. Every

corner belongs to exactly one vertex, so $4N = \sum_i v_i$. Such a mesh is thus an existence certificate for its own signature, and this is the form in which all constructions in this paper are given.

We fix the combinatorial model once. The *darts* are the pairs (face, side), encoded as $4f + s$ with $s \in \{0, 1, 2, 3\}$: dart $4f + s$ is side s of square f , traversed counterclockwise. Then $\sigma(4f + s) = 4f + (s + 1 \bmod 4)$ is the face rotation, the gluing is a fixed-point-free involution α on darts, and the vertices are the orbits of $\nu = \alpha \circ \sigma^{-1}$, the orbit through a dart consisting of the darts whose tail is that vertex, so that its length is the valence. Orienting side s so that its outward normal points in direction s , gluing side s of f to side t of h makes the chart transition from f to h the rotation

$$r = (t - s + 2) \bmod 4,$$

which is antisymmetric under swapping the two darts, as it must be. In particular a gluing of two *opposite* sides, $t = s + 2$, is a translation, $r = 0$, while a gluing of a side to itself in reverse is a half-turn. Finally ρ is read off the dual graph: the dual 1-skeleton is a deformation retract of M minus the vertices, so its cycles generate $H_1(M \setminus C)$, and ρ of a cycle is the sum of the rotations it crosses.

2.3 Results used as black boxes

The following are used without proof. For $g = 0$ the classification uses Troyanov (S6), through Proposition 3.7; for $g \geq 1$ it uses the Gendron–Tahar classification of primitive k -differential strata, (S7). Beyond routine topology those are the only two that matter.

- (S1) A flat cone metric on a closed surface admits a geodesic triangulation whose vertex set contains the cone points.
- (S2) Two triangulations of a compact surface admit a common refinement.
- (S3) $\mathrm{Sp}(2g, \mathbb{Z})$ acts transitively on primitive vectors of $\mathbb{Z}^{2g}$.
- (S4) $\mathrm{Sp}(2g, \mathbb{Z}) \rightarrow \mathrm{Sp}(2g, \mathbb{Z}/N)$ is surjective.
- (S5) $\mathrm{MCG}(\Sigma_{g,1}, \partial) \rightarrow \mathrm{Sp}(2g, \mathbb{Z})$ is surjective.
- (S6) (Troyanov [15]) On a closed surface, for $n \geq 3$ distinct points and angles $\theta_i > 0$ satisfying Gauss–Bonnet there is a flat cone metric with those angles at those points.
- (S7) (Gendron–Tahar [6, Th. 1.4]) Let $g \geq 1$, let $k \geq 1$, and let $\mu = (a_1, \dots, a_n)$ be a partition of $k(2g - 2)$ with every $a_i > -k$. The stratum $\Omega^k \mathcal{M}_g(\mu)$ of *primitive* k -differentials of profile μ , those that are not powers of k' -differentials for $1 \leq k' < k$, is empty if and only if
 - (i) $g = 1$ and $\mu = (1, -1)$;
 - (ii) $g = 1$ and $\mu = \emptyset$ and $k \geq 2$;
 - (iii) $g = 2$, $k = 2$ and $\mu = (4)$ or $\mu = (3, 1)$.

The hypothesis $g \geq 1$ is the scope of the section of [6] in which Theorem 1.4 appears, and it cannot be dropped. For $g = 0$, $k = 4$ and $\mu = (-2, -2, -2, -2)$, which is admissible since $\sum_i a_i = -8 = k(2g - 2)$ and $a_i > -4$, the only 4-differential with that divisor on $\mathbb{P}^1$ is $c(dz)^4 / \prod_i (z - p_i)^2$, the square of $\sqrt{c}(dz)^2 / \prod_i (z - p_i)$, so that stratum is empty although it matches no case above. Genus 0 is handled separately, by Proposition 3.7.

For $g \geq 1$ this one statement supplies every non-emptiness input the classification needs, for all three values of $k = 4/d$ at once. Its case (iii) for $k = 2$ is the classical theorem of Masur and

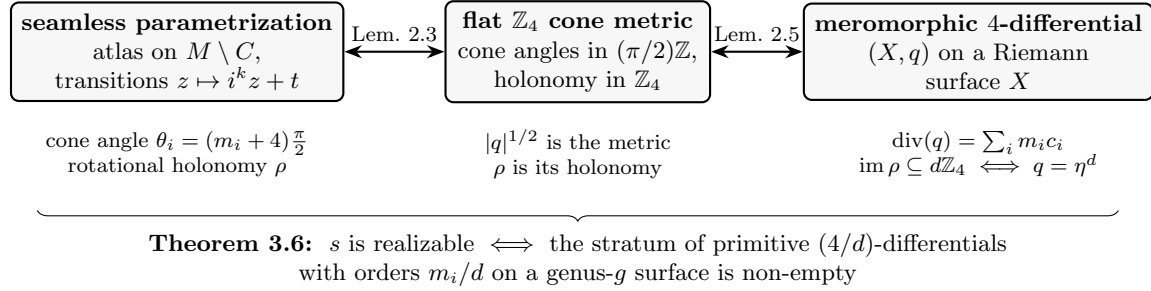


Figure 2: The correspondences of §2.4. Reading left to right, a seamless parametrization is a flat cone metric with holonomy in $\mathbb{Z}_4$ (Lemma 2.3), and such a metric is $|q|^{1/2}$ for a meromorphic 4-differential q whose zeros and poles are the cones (Lemma 2.5). Under this correspondence $\text{im } \rho$ measures primitivity: q is a d -th power exactly when $\text{im } \rho \subseteq d\mathbb{Z}_4$, so $\text{im } \rho = d\mathbb{Z}_4$ says that $q = \eta^d$ with η *primitive* (Lemma 2.6). Combined with the Reduction Lemma this turns the question into one about strata, which §3 answers.

Smillie [13], and its cases (i) and (ii) are reproved independently here as Proposition 3.8. What we take purely on trust is that the list stops there.

2.4 The dictionary

This subsection proves the three correspondences summarized in Figure 2.

Lemma 2.3 (Seamless parametrization = flat $\mathbb{Z}_4$ cone metric). *Seamless parametrizations of (M, C) are the same thing as flat cone metrics on M with cone points contained in C , cone angles in $(\pi/2)\mathbb{Z}$, and rotational holonomy contained in $\mathbb{Z}_4$. The signature corresponds on both sides.*

Proof. Given a seamless atlas, pull back the euclidean metric; this is well defined since the transitions $z \mapsto i^k z + t$ are isometries. Parallel transport along a loop γ is the rotation part $i^{k(\gamma)}$ of the composite transition, and rotation parts multiply in the abelian group μ_4 , so $\gamma \mapsto k(\gamma)$ is the rotational holonomy $H_1(M \setminus C; \mathbb{Z}) \rightarrow \mathbb{Z}_4$.

Conversely, developing maps on simply connected subsets of $M \setminus C$ give an atlas whose transitions are orientation-preserving euclidean isometries and are local isometries, hence locally injective. The rotation part around any loop is the rotational holonomy, which lies in $\mathbb{Z}_4$ by hypothesis, so on a good cover the transitions may be taken of the form $z \mapsto i^k z + t$. The holonomy around γ_i is rotation by θ_i , forcing $\rho(\gamma_i) = m_i \bmod 4$ as in Definition 2.1. $\square$

Lemma 2.4 (Realizability is topological). (a) *The set of realizable signatures is invariant under $\text{MCG}(M, C)$.* (b) *A signature s is realizable if and only if there is a flat cone metric on M with cone angles $(m_i + 4)\pi/2$ at n distinct points whose signature lies in the $\text{MCG}(M, C)$ -orbit of s .*

Proof. (a) If $\varphi \in \text{MCG}(M, C)$ and $\{U_a, f_a\}$ realizes s , then $\{\varphi^{-1}(U_a), f_a \circ \varphi\}$ is again a seamless atlas with the same transitions, and its signature is $\varphi^* s$. Any mapping class is represented by a PL homeomorphism, so the atlas is again PL on a refinement of a triangulation.

(b) Necessity is Lemma 2.3. For sufficiency, let μ be a flat cone metric with cone points $p_1, \dots, p_n$ and signature in the orbit of s . A homeomorphism ψ with $\psi(c_i) = p_{\sigma(i)}$ matching orders makes $\psi^* \mu$ have cone set exactly C , and by (a) we may assume its signature is s ; Lemma 2.3

then gives a seamless atlas. To make it PL on a refinement of the given triangulation T : by (S1) the metric has a geodesic triangulation T'' containing the cone points among its vertices, and the developing charts are affine on each of its triangles; by (S2) a common refinement of T and T'' works. $\square$

Two consequences are worth isolating, because they are what makes the rest possible: the conformal structure is free, and so are the cone positions. This is the precise sense in which the question differs from the fixed-conformal-structure criterion discussed in §6.2.

Lemma 2.5 (Flat $\mathbb{Z}_4$ cone metric = meromorphic 4-differential). *Flat cone metrics on M with cone angles in $(\pi/2)\mathbb{Z}$ and rotational holonomy in $\mathbb{Z}_4$ are the same thing as pairs (X, q) with X a Riemann surface homeomorphic to M and q a meromorphic section of K_X^4 with $\text{div}(q) = \sum_i m_i c_i$, $m_i > -4$. The metric is $|q|^{1/2}$, the cone angle at c_i is $(m_i + 4)\pi/2$, and the rotational holonomy is the monodromy of the local fourth roots of q .*

Proof. Let μ be such a metric, with developing charts z_a and transitions $z_b = i^k z_a + t$. Then $(dz_b)^4 = i^{4k}(dz_a)^4 = (dz_a)^4$, so the local 4-differentials $(dz_a)^4$ agree on overlaps and define a nowhere-zero holomorphic section q of K^4 on $M \setminus C$ with $|q|^{1/2} = \mu$. Near a cone of angle $(m+4)\pi/2$ the metric is isometric to the standard cone, developed in a holomorphic coordinate w by $z = \frac{4}{m+4}w^{(m+4)/4}$; hence

$$q = (dz)^4 = w^m(dw)^4,$$

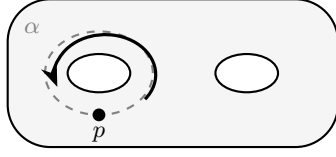
so q extends meromorphically across c_i with order exactly m_i , and $m_i > -4$ is precisely positivity of the cone angle. Since $\deg K_X^4 = 4(2g - 2)$, the degree of $\text{div}(q)$ reproduces Gauss–Bonnet. Conversely, given (X, q) , the metric $|q|^{1/2}$ is flat away from $\text{div}(q)$ with the stated cone angles; a local primitive $z = \int q^{1/4}$ is a developing chart, two choices differ by $z \mapsto i^k z + t$, and so the rotational holonomy is the monodromy of $q^{1/4}$ and lies in $\mu_4 = \mathbb{Z}_4$. $\square$

Call a k -differential *primitive* if it is not an e -th power for any $e > 1$ dividing k ; every abelian differential is primitive.

Lemma 2.6 (Primitivity is measured by $\text{im } \rho$). *Let q be as in Lemma 2.5, with rotational holonomy ρ , and let $e \mid 4$. Then $q = \eta^e$ for a meromorphic section η of $K^{4/e}$ if and only if $\text{im } \rho \subseteq e\mathbb{Z}_4$. Consequently, if d generates $\text{im } \rho$ then $q = \eta^d$ for a primitive $(4/d)$ -differential η with $\text{div}(\eta) = \sum_i (m_i/d)c_i$; and conversely η^d has rotational holonomy with image exactly $d\mathbb{Z}_4$.*

Proof. Work locally, $q = f(dz)^4$. A fourth root of q is $f^{1/4}dz$, and by Lemma 2.5 the holonomy ρ is the monodromy character of the multivalued $f^{1/4}$, valued in μ_4 . An e -th root of q is $h(dz)^{4/e}$ with $h^e = f$, that is $h = f^{1/e} = (f^{1/4})^{4/e}$, which is single valued if and only if $\rho(\gamma)^{4/e} = 1$ for all γ ; additively this reads $(4/e)\rho(\gamma) = 0$, that is $\rho(\gamma) \in e\mathbb{Z}_4$. This also forces $e \mid m_i$, since $\rho(\gamma_i) = m_i \bmod 4$ and $e \mid 4$.

Now let d generate $\text{im } \rho$. By the above there is η with $\eta^d = q$ and $\text{div}(\eta) = \text{div}(q)/d$. Its own holonomy $\rho_\eta: H_1 \rightarrow \mathbb{Z}_{4/d}$ satisfies $\rho = d \cdot \rho_\eta$ under $\mathbb{Z}_{4/d} \cong d\mathbb{Z}_4$, so $\text{im } \rho_\eta$ is all of $\mathbb{Z}_{4/d}$; by the first claim applied to η this says exactly that η has no e -th root for $e > 1$ dividing $4/d$. The converse is the same computation read backwards. $\square$



push the cone p once around α

$$P(\alpha)^*\rho = \rho + m_p \langle \cdot, \alpha \rangle$$

ρ is unchanged on every γ_i ,
and shifts by m_p along each loop meeting α once

so pushes generate translations by $D \cdot H^1(M; \mathbb{Z}_4)$

Figure 3: Point pushing (Lemma 3.1). Dragging a cone of order m_p once around a simple closed curve α is a mapping class fixing C pointwise, and it changes the holonomy by m_p times the intersection pairing with α . The loops γ_i around the cones are left alone, so the cone angles do not move. Ranging over p and α , these pushes translate $A(m)$ by the subgroup $D \cdot H^1(M; \mathbb{Z}_4)$, where $D = \langle m_1, \dots, m_n \rangle$. When some m_p is odd we have $D = \mathbb{Z}_4$ and the pushes alone act transitively, which is Corollary 3.5.

3 The classification

3.1 The mapping class group action

Throughout this subsection $g \geq 1$ and the orders m are fixed. Recall that $A(m)$ is a coset of $H^1(M; \mathbb{Z}_4) \cong \mathbb{Z}_4^{2g}$ and that $D = \langle m_1, \dots, m_n \rangle$.

Lemma 3.1 (Point pushing). *Let $p \in C$ and let $\alpha \in H_1(M; \mathbb{Z})$ be represented by a simple closed curve through p , and let $P(\alpha) \in \text{MCG}(M, C)$ be the corresponding point-pushing map. Then*

$$P(\alpha)_*x = x + \langle x, \alpha \rangle \gamma_p \quad \text{for } x \in H_1(M \setminus C; \mathbb{Z}), \quad \text{hence} \quad P(\alpha)^*\rho = \rho + m_p \langle \cdot, \alpha \rangle.$$

As p and α vary these moves generate exactly the translations of $A(m)$ by $D \cdot H^1(M; \mathbb{Z}_4)$, and they fix every puncture value. See Figure 3.

Proof. $P(\alpha)$ is the composition $T_{\alpha_L} T_{\alpha_R}^{-1}$ of Dehn twists along the two boundary curves of an annulus neighbourhood of the curve, with p in the annulus between them. On $H_1(M \setminus C)$ these are transvections, and since the two curves are disjoint, $\langle \alpha_L, \alpha_R \rangle = 0$ and the twists commute, so

$$P(\alpha)_*x = x + \langle x, \alpha_L \rangle \alpha_L - \langle x, \alpha_R \rangle \alpha_R.$$

Both boundary curves are disjoint from C and homologous to α in M , so their intersection numbers with x agree and equal $\langle x, \alpha \rangle$; and their difference in $H_1(M \setminus C)$ is γ_p , because the annulus between them contains exactly the puncture p . Hence $P(\alpha)_*x = x + \langle x, \alpha \rangle \gamma_p$ and $(P(\alpha)^*\rho)(x) = \rho(x) + m_p \langle x, \alpha \rangle$. Puncture values are unchanged because $\langle \gamma_i, \alpha \rangle = 0$. Finally $\alpha \mapsto \langle \cdot, \alpha \rangle$ is Poincaré duality, so already for α in a symplectic basis these functionals span $m_p H^1(M; \mathbb{Z}_4)$; summing over p gives $D \cdot H^1(M; \mathbb{Z}_4)$, and no more, since every such move is of this form. $\square$

Lemma 3.2 (Symplectic transitivity). *Let $g \geq 1$, $N \geq 1$. For $v \in (\mathbb{Z}/N)^{2g}$ set $\text{cont}(v) = \gcd(N, v_1, \dots, v_{2g})$. Then $\text{Sp}(2g, \mathbb{Z}/N)$ preserves cont and acts transitively on each of its level sets.*

Proof. Invariance is clear, as S is invertible over $\mathbb{Z}/N$. For transitivity, let $\text{cont}(v) = c$, write $N = cN'$ and $v = ca$ with $a \in (\mathbb{Z}/N')^{2g}$ and $\gcd(N', a_1, \dots, a_{2g}) = 1$.

Step 1: lift a to a primitive integer vector. Adjusting one coordinate does not suffice: for $N' = 5$ and $a = (2, 0, \dots, 0)$ every lift $(2 + 5k, 0, \dots, 0)$ has content $|2 + 5k| \neq 1$. So we move two, which is possible as $2g \geq 2$.

Choose a lift $\tilde{a}$ with $\tilde{a}_1 \neq 0$ (add a multiple of N' if needed) and set $D = \gcd(\tilde{a}_1, \tilde{a}_3, \dots, \tilde{a}_{2g}) > 0$, which omits $\tilde{a}_2$ and is therefore fixed once and for all. Let $p \mid D$ be prime. If $p \mid N'$ then $p \nmid \tilde{a}_2$, since otherwise p would divide N' and every $\tilde{a}_i$, against $\gcd(N', a) = 1$; so $\tilde{a}_2 + N't$ is prime to p for every t . If $p \nmid N'$, then N' is invertible mod p and $t \equiv (1 - \tilde{a}_2)(N')^{-1}$ makes $\tilde{a}_2 + N't \equiv 1$. Choosing such a t simultaneously for the finitely many primes of the second kind, by the Chinese Remainder Theorem, gives $\gcd(D, \tilde{a}_2 + N't) = 1$, hence $\gcd(\tilde{a}_1, \tilde{a}_2 + N't, \tilde{a}_3, \dots, \tilde{a}_{2g}) = 1$: a primitive lift.

Step 2. By (S3) there is $\tilde{S} \in \mathrm{Sp}(2g, \mathbb{Z})$ with $\tilde{S}\tilde{a} = e_1$, hence $\tilde{S}(c\tilde{a}) = ce_1$.

Step 3. By (S4) the reduction S of $\tilde{S}$ lies in $\mathrm{Sp}(2g, \mathbb{Z}/N)$ and $Sv = ce_1$. Every v of content c maps to the same ce_1 , so the level set is one orbit. $\square$

Lemma 3.3 (A base point with full symplectic stabilizer). *There exist $\rho_0 \in A(m)$ and a subgroup $H \leq \mathrm{MCG}(M, C)$ fixing C pointwise such that H fixes ρ_0 and the image of H in $\mathrm{Aut}(H^1(M; \mathbb{Z}_4))$ is all of $\mathrm{Sp}(2g, \mathbb{Z}_4)$.*

Proof. Choose an embedded closed disk $D_0 \subset M$ containing C in its interior and let $\Sigma = M \setminus \mathrm{int}(D_0)$, a genus- g surface with one boundary circle. Choose a symplectic basis $a_1, b_1, \dots, a_g, b_g$ of $H_1(M; \mathbb{Z})$ represented by simple closed curves in the interior of Σ . Define ρ_0 by $\rho_0(\gamma_i) = m_i$ and $\rho_0(a_j) = \rho_0(b_j) = 0$; this is well defined because the only relation among the generators is $\sum_i \gamma_i = 0$ and $\sum_i m_i \equiv 0$. Note that ρ_0 vanishes on the image of $H_1(\Sigma; \mathbb{Z}) \rightarrow H_1(M \setminus C; \mathbb{Z})$, which is generated by the a_j, b_j and by $[\partial\Sigma] = \sum_i \gamma_i$.

Let H be the image of $\mathrm{MCG}(\Sigma, \partial\Sigma) \rightarrow \mathrm{MCG}(M, C)$, extending by the identity on D_0 . Every $\varphi \in H$ is the identity near C , so $\varphi_*\gamma_i = \gamma_i$; and for any 1-cycle x the class $\varphi_*x - x$ lies in the image of $H_1(\Sigma)$, since φ is the identity outside Σ and near $\partial\Sigma$, so x and φ_*x agree outside Σ and their arcs inside have the same endpoints. Hence $(\varphi^*\rho_0)(x) = \rho_0(x) + \rho_0(\varphi_*x - x) = \rho_0(x)$. Finally $\mathrm{MCG}(\Sigma, \partial\Sigma) \rightarrow \mathrm{Sp}(2g, \mathbb{Z})$ is surjective by (S5) and $\mathrm{Sp}(2g, \mathbb{Z}) \rightarrow \mathrm{Sp}(2g, \mathbb{Z}_4)$ is surjective by (S4). $\square$

With ρ_0 fixed, identify $A(m)$ with $\mathbb{Z}_4^{2g}$ by $\rho = \rho_0 + v$. Then H acts linearly, $v \mapsto Sv$, and point pushing acts by translations $v \mapsto v + w$, $w \in D^{2g}$, and

$$\mathrm{im}(\rho_0 + v) = \langle D, v_1, \dots, v_{2g} \rangle = \gcd(d_D, v_1, \dots, v_{2g}) \mathbb{Z}_4, \quad (1)$$

where d_D generates D .

Theorem 3.4 (Reduction Lemma). *Let $g \geq 1$ and fix the orders m . Two signatures with orders m lie in the same $\mathrm{MCG}(M, C)$ -orbit if and only if they have the same $\mathrm{im} \rho$. Hence the orbits with orders m are in bijection with the subgroups of $\mathbb{Z}_4$ containing D : there are 1, 2 or 3 of them according as $D = \mathbb{Z}_4$, $D = 2\mathbb{Z}_4$, $D = 0$.*

Proof. If $\rho' = \rho \circ \varphi_*$ then $\mathrm{im} \rho' = \rho(\varphi_* H_1) = \rho(H_1) = \mathrm{im} \rho$, so the invariant is constant on orbits.

Conversely let $v, w \in \mathbb{Z}_4^{2g}$ with $\gcd(d_D, v) = \gcd(d_D, w)$, by (1). Reduce modulo D : since $d_D \mid 4$ we have $\mathbb{Z}_4/D = \mathbb{Z}/d_D$, and the point-pushing translations by D^{2g} act transitively on each fibre of $\mathbb{Z}_4^{2g} \rightarrow (\mathbb{Z}/d_D)^{2g}$. The reductions $\bar{v}, \bar{w}$ have equal content in $(\mathbb{Z}/d_D)^{2g}$, so by Lemma 3.2 with $N = d_D$ there is $\bar{S} \in \mathrm{Sp}(2g, \mathbb{Z}/d_D)$ with $\bar{S}\bar{v} = \bar{w}$; by (S4) it lifts to $S \in \mathrm{Sp}(2g, \mathbb{Z}_4)$, which by Lemma 3.3 is realized by some $\varphi \in H$. Applying φ and then a point-pushing translation carries v to w . $\square$

Corollary 3.5. *If some cone angle is an odd multiple of $\pi/2$, in particular if some cone has angle $3\pi/2$ or $5\pi/2$, then $D = \mathbb{Z}_4$, all signatures with those cone angles form a single MCG-orbit, and realizability does not depend on ρ .*

Proof. m_i odd gives $D = \mathbb{Z}_4$ and $d_D = 1$, so $(\mathbb{Z}/d_D)^{2g}$ is trivial and Theorem 3.4 leaves one orbit. $\square$

Mechanically, pushing a cone of odd order once around a handle shifts the holonomy along the dual loop by an odd amount, so repeated pushes reach every value. This is where the gcd hypothesis of [14] comes from, and it is also the point at which less turns out to be needed: their rerouting must reproduce turning numbers in $\frac{1}{4}\mathbb{Z}$ exactly and so asks for $\gcd_{\mathbb{Z}}(m_i) = 1$, while realizability sees only the class of ρ in $\mathbb{Z}_4$ and so asks for $\langle m_i \rangle = \mathbb{Z}_4$. See §1.3(ii).

3.2 Realizability is non-emptiness of a stratum

Theorem 3.6. *Let $s = (m, \rho)$ be a holonomy signature on (M, C) , $g = \text{genus}(M)$, and let d generate $\text{im } \rho$. Then s is realizable if and only if the stratum of primitive $(4/d)$ -differentials with orders $(m_1/d, \dots, m_n/d)$ on a genus- g surface is non-empty.*

Proof. If s is realizable, Lemmas 2.3 and 2.5 give (X, q) with $\text{div}(q) = \sum m_i c_i$ and holonomy ρ , and Lemma 2.6 extracts a primitive $(4/d)$ -differential η with orders m_i/d .

Conversely let η be a primitive $(4/d)$ -differential with orders m_i/d on a genus- g Riemann surface X , with zeros and poles at $p_1, \dots, p_n$, and put $q = \eta^d$. By Lemma 2.6 the holonomy of q has image exactly $d\mathbb{Z}_4$, and $\text{div}(q) = \sum_i m_i p_i$, so by Lemma 2.5 the metric $|q|^{1/2}$ is a flat cone metric on X with the prescribed cone angles. Transport it to M by an orientation-preserving homeomorphism $X \rightarrow M$ carrying p_i to $c_{\sigma(i)}$ for a permutation σ matching equal orders; the resulting signature s' has the same multiset of orders as s and $\text{im } \rho' = d\mathbb{Z}_4 = \text{im } \rho$. For $g \geq 1$, Theorem 3.4 puts s' in the orbit of s ; for $g = 0$ we have $H^1(M; \mathbb{Z}_4) = 0$, so $A(m)$ is a single point and $s' = s$ after relabelling C . Either way Lemma 2.4(b) gives realizability of s . $\square$

The problem is therefore *equivalent* to a stratum non-emptiness question. For $d = 4$ these are strata of abelian differentials, for $d = 2$ strata of primitive quadratic differentials, and for $d = 1$ strata of primitive 4-differentials.

This is the point at which the question meets a developed theory. Strata of abelian and quadratic differentials, their non-emptiness and their connected components, are classical [13, 8, 9]; existence with prescribed singularities has since been settled uniformly in k by Gendron and Tahar [6, 4, 5], whose Theorem 1.4 is quoted as (S7) and is the one external input the classification uses. The construction of §4 is independent of it, and over the genus-two quadratic strata it reproduces that theorem's exceptional list exactly.

3.3 Genus zero and genus one

Proposition 3.7. *Every Gauss–Bonnet-admissible holonomy signature on the sphere is realizable.*

Proof. First, $n \geq 3$: Gauss–Bonnet gives $\sum_i m_i = -8$ while $m_i > -4$ forces $\sum_i m_i > -4n$. Second, ρ is determined by m , because $H_1(S^2 \setminus C; \mathbb{Z})$ is generated by the γ_i ; in particular any flat cone metric on S^2 with cone angles in $(\pi/2)\mathbb{Z}$ automatically has holonomy in $\mathbb{Z}_4$, and its signature is the given one. Third, such a metric exists by (S6). Now apply Lemma 2.4(b). $\square$

For genus 1 the answer comes from Abel–Jacobi, in a form that treats all three values of d at once.

Proposition 3.8. *Let $k \in \{1, 2, 4\}$ and let $\mu = (\mu_1, \dots, \mu_n)$ be integers with $\mu_i > -k$ and $\sum_i \mu_i = 0$. Let n_0 be the number of nonzero μ_i . The stratum of primitive k -differentials with orders μ on a genus-1 surface is non-empty if and only if*

- (i) $n_0 = 0$ and $k = 1$; or
- (ii) $n_0 = 2$, in which case the nonzero orders are $(\mu, -\mu)$, and $|\mu| \geq 2$; or
- (iii) $n_0 \geq 3$.

($n_0 = 1$ cannot occur, since the orders sum to zero.)

Proof. On $E = \mathbb{C}/\Lambda$ the canonical bundle is trivialized by dz , so every meromorphic k -differential is $q = f(dz)^k$ with f meromorphic and $\operatorname{div}(q) = \operatorname{div}(f)$. By Abel–Jacobi a degree-zero divisor $\sum_i \mu_i c_i$ is principal if and only if

$$\sum_i \mu_i c_i = 0 \quad \text{in the group law of } E. \quad (2)$$

By Lemma 2.6, q is primitive if and only if for every $e > 1$ dividing k with $e \mid \mu_i$ for all i ,

$$\sum_i (\mu_i/e) c_i \neq 0. \quad (3)$$

Necessity. If $n_0 = 0$ then f has no zeros or poles, hence is constant, and $q = c(dz)^k = (c^{1/k} dz)^k$ is a k -th power; for $k > 1$ this contradicts primitivity, while for $k = 1$ it is the flat torus with marked points. If $n_0 = 2$ the nonzero orders are $(\mu, -\mu)$ at distinct $c_1 \neq c_2$, and (2) reads $\mu(c_1 - c_2) = 0$ with $c_1 - c_2 \neq 0$, that is E has a nonzero μ -torsion point; for $|\mu| = 1$ there is none.

Sufficiency. Let e^* be the largest divisor of k dividing every μ_i , put $\nu = \mu/e^*$, and pick $T \in E$ of exact order e^* (so $T = 0$ if $e^* = 1$). Since the divisors of $k \in \{1, 2, 4\}$ form a chain, every $e > 1$ occurring in (3) divides e^* . It therefore suffices to find *distinct* points with $\sum_i \nu_i c_i = T$: then $\sum_i \mu_i c_i = e^* T = 0$, giving (2), while for each relevant e , $\sum_i (\mu_i/e) c_i = (e^*/e) T$ has exact order e and so is nonzero, giving (3).

If $n_0 = 2$ then $\nu = (\nu, -\nu)$ and we need $\nu(c_1 - c_2) = T$. For $e^* > 1$, multiplication by ν is surjective, so choose u with $\nu u = T$; then $u \neq 0$ as $T \neq 0$. For $e^* = 1$ we have $T = 0$ and $|\nu| = |\mu| \geq 2$, so take u a nonzero $|\nu|$ -torsion point, which exists as $E[|\nu|] \cong (\mathbb{Z}/|\nu|)^2$. Put $c_1 = u$, $c_2 = 0$.

If $n_0 \geq 3$, let $\Psi: E^{n_0} \rightarrow E$ be $\Psi(c) = \sum_i \nu_i c_i$ over the nonzero indices. Some $\nu_i \neq 0$ and multiplication by it is surjective, so Ψ is; the identity component K^0 of $\ker \Psi$ is an abelian subvariety of dimension $n_0 - 1 \geq 2$, and $\Psi^{-1}(T)$ contains the irreducible positive-dimensional coset $x_0 + K^0$. For $i \neq j$ among the nonzero indices, $c_i - c_j$ is non-constant on K^0 : if $\nu_i + \nu_j \neq 0$ the one-parameter subgroup $(c_i, c_j) = (\nu_j t, -\nu_i t)$ lies in K^0 and there $c_i - c_j = (\nu_i + \nu_j)t$; otherwise take a third nonzero index l , available since $n_0 \geq 3$, and $(c_i, c_l) = (\nu_l t, -\nu_i t)$, on which $c_i - c_j = \nu_l t$. Translation by x_0 preserves non-constancy, so each diagonal $\{c_i = c_j\}$ meets $x_0 + K^0$ in a proper closed subset; by irreducibility their finite union is not everything, so some point of $\Psi^{-1}(T)$ has distinct coordinates. The marked points do not appear in Ψ and go anywhere else on E . $\square$

Corollary 3.9 (The torus). *A holonomy signature on the torus is realizable if and only if it is not one of*

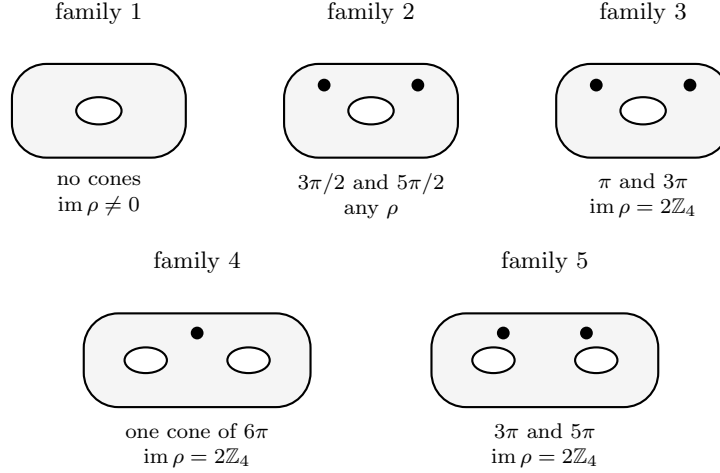


Figure 4: The five unrealizable holonomy signatures of Table 1; dots mark cones. Families 1 to 3 live on the torus and are proved in Proposition 3.8; family 2 is the exception already known to [14, 7]. Families 4 and 5 appear to be new in this setting: a genus-two surface whose cross field splits globally into two line fields, the middle column of Figure 1, cannot be quadrangulated with a single cone of angle 6π , or with a pair of cones of angles 3π and 5π . Every other Gauss–Bonnet-admissible signature, in every genus, is realizable.

- (1) *no cones, with $\text{im } \rho \neq 0$;*
- (2) *exactly two cones, of angles $3\pi/2$ and $5\pi/2$;*
- (3) *exactly two cones, of angles π and 3π , with $\text{im } \rho = 2\mathbb{Z}_4$.*

Proof. Apply Theorem 3.6 with $g = 1$, $k = 4/d$, $\mu = m/d$ and read Proposition 3.8. The empty cases are: $n_0 = 0$ with $k > 1$, that is no cones and $d \neq 4$, which is (1); and $n_0 = 2$ with $|\mu| = 1$, that is two cones with orders $(m, -m)$ and $|m| = d$. Since $d \in \{1, 2, 4\}$ and $m > -4$, the possibilities are $|m| = d = 1$, giving cone angles $3\pi/2$ and $5\pi/2$, and then $D = \mathbb{Z}_4$ forces $d = 1$, so the case is independent of ρ , which is (2); and $|m| = d = 2$, giving cone angles π and 3π with $\text{im } \rho = 2\mathbb{Z}_4$, which is (3). The value $|m| = d = 4$ would require an order -4 , which is excluded. $\square$

Case (2) is the exception of [14] and the non-existence of a 3, 5-quadrangulation of the torus of [7]; here it comes out in one line, as the absence of a nonzero 1-torsion point. Case (3) is the empty stratum $Q(1, -1)$ of [13]. It is worth contrasting (3) with the *realizable* signature “two cones of angles π and 3π , some odd holonomy”: the cone angles alone do not decide.

3.4 The five exceptions

Theorem 3.10. *A holonomy signature (g, m, ρ) is realizable if and only if it is not one of the five families of Table 1. For $g \geq 1$ these five are precisely the exceptional list of (S7), read through Theorem 3.6; for $g = 0$ there are none, by Proposition 3.7.*

Proof. For $g = 0$, Proposition 3.7 says every Gauss–Bonnet-admissible signature is realizable, and Table 1 lists none; (S7) is neither available nor needed there. Assume $g \geq 1$ from now on.

	genus	cone angles	holonomy	reduced stratum	status
1	1	none	$\text{im } \rho \neq 0$	none	Prop. 3.8
2	1	$3\pi/2, 5\pi/2$	any	primitive 4-diff.	Prop. 3.8
3	1	$\pi, 3\pi$	$\text{im } \rho = 2\mathbb{Z}_4$	$Q(1, -1)$	Prop. 3.8
4	2	6π	$\text{im } \rho = 2\mathbb{Z}_4$	$Q(4)$	(S7)(iii)
5	2	$3\pi, 5\pi$	$\text{im } \rho = 2\mathbb{Z}_4$	$Q(1, 3)$	(S7)(iii)

Table 1: The unrealizable holonomy signatures.

By Theorem 3.6, s is realizable exactly when the stratum of primitive k -differentials with orders $\mu = (m_1/d, \dots, m_n/d)$ is non-empty, where $k = 4/d$. The hypotheses of (S7) are met: $\sum_i m_i/d = 4(2g - 2)/d = k(2g - 2)$, and $m_i > -4$ with $d \mid m_i$ gives $m_i/d > -4/d = -k$. So the unrealizable signatures are exactly the exceptional list of (S7), pulled back along $\mu = m/d$. We unwind its three cases.

(i) $g = 1$, $\mu = (1, -1)$, any k . Then $m = (d, -d)$, cone angles $(d + 4)\pi/2$ and $(4 - d)\pi/2$. For $d = 1$: angles $5\pi/2$ and $3\pi/2$, family 2, the exception of [14, 7]. For $d = 2$: angles 3π and π with $\text{im } \rho = 2\mathbb{Z}_4$, family 3. For $d = 4$ the datum $m = (4, -4)$ violates $m_i > -4$ and does not arise.

(ii) $g = 1$, $\mu = \emptyset$, $k \geq 2$. No cones, and $k \geq 2$ says $d \in \{1, 2\}$, that is $\text{im } \rho \neq 0$, family 1. For $k = 1$, that is $d = 4$, the stratum is the flat torus and is non-empty; this is why family 1 carries the hypothesis $\text{im } \rho \neq 0$.

(iii) $g = 2$, $k = 2$, $\mu = (4)$ or $(3, 1)$. Here $d = 2$, so $\text{im } \rho = 2\mathbb{Z}_4$ and $m = 2\mu$. From $\mu = (4)$: $m = (8)$, one cone of angle 6π , family 4. From $\mu = (3, 1)$: $m = (6, 2)$, cones of angles 5π and 3π , family 5.

Those are the five rows of Table 1, and for $g \geq 1$ (S7) says there is nothing else, for any d . $\square$

The five are drawn in Figure 4. Families 4 and 5 appear to be new in this setting. In the language of cross fields: *a genus-two surface carrying a cross field whose holonomy along every homology loop is a multiple of π , equivalently one that splits globally into a pair of line fields, cannot be quadrangulated if its singularities are a single cone of angle 6π , or a pair of cones of angles 3π and 5π* . Both have even cone orders, so they lie outside the gcd condition of [14] and contradict nothing in it; they populate the gap it left, and §5 works through that gap.

4 Explicit square-tiled realizations

Theorem 3.10 settles non-emptiness by citation. This section does something a citation does not: it gives an explicit constructive procedure in the square-tiled model of Remark 2.2, where a mesh is its own certificate, and it is where every explicit witness in this paper comes from.

4.1 Two local surgeries

Lemma 4.1 (Splitting Lemma). *Let M be a closed quad mesh, v a vertex of valence w , and e a loop at v , that is an edge both of whose endpoints are v . Let γ be the angular gap of e : the number of corners at v strictly between the two ends of e , counted counterclockwise from the outgoing end. Cut M along e and glue one unit square into the slit as follows (Figure 5). In*

the dart model of Remark 2.2, let x be a dart of e and $\alpha(x)$ its partner, let the new square have darts $y_0, \dots, y_3$ with y_s its side s , and set

$$\alpha'(x) = y_2, \quad \alpha'(\alpha(x)) = y_0, \quad \alpha'(y_1) = y_3,$$

leaving α unchanged elsewhere. The two halves of the cut edge are thus attached to opposite sides of the new square, which is the half-turn and is what makes the move split rather than merely subdivide, while the two leftover sides y_1, y_3 , also opposite, are glued to each other by a translation. The result M' is a closed quad mesh with

- (i) one more square;
- (ii) the same genus;
- (iii) the same vertices, except that v is replaced by two vertices of valences $\gamma + 2$ and $w - \gamma + 2$;
- (iv) the same $\text{im } \rho$.

Proof. Cutting. e is a geodesic segment from v to itself, so cutting along it opens a slit: the two copies of e become boundary, and the cyclic order of the w corners at v is severed at the two ends of e , leaving two boundary arcs with γ and $w - \gamma$ corners.

Filling. Sides y_2, y_0 of the new square are glued to the two copies of e , both unit geodesics, and attaching them to opposite sides is the half-turn; then y_1, y_3 are glued to each other, leaving no boundary. That last gluing is a *translation*: by Remark 2.2 its rotation is $(3 - 1 + 2) \equiv 0$. This is what the lemma turns on and is not optional; gluing y_1 to y_3 by a half-turn would put 2 into $\text{im } \rho'$ and destroy (iv) whenever $\text{im } \rho = 0$.

Vertices. The square contributes two corners to each end of the slit, giving valences $\gamma + 2$ and $w - \gamma + 2$. No other vertex is touched, since the surgery happens inside the closed star of v . This is where the hypothesis that e is a *loop* is used, and it cannot be dropped: cutting an edge with an endpoint $u \neq v$ adds corners at u .

Genus. F and V each rise by one and E by two, the cut edge being replaced by the two sides glued to the square together with the self-glued edge; so χ is unchanged.

Holonomy. Old dual cycles survive, so $\text{im } \rho \subseteq \text{im } \rho'$, and their values survive too: if the old gluing joined side s to side t , with rotation $r_{\text{old}} = (t - s + 2)$, the two new edges carry $r_1 = (2 - s + 2) \equiv -s$ and $r_2 = (t - 0 + 2) \equiv t + 2$, summing to r_{old} . Conversely the new dual graph is the old one with that edge subdivided, which does not change the cycle space, plus one loop at the new node, dual to $\{y_1, y_3\}$ and of rotation 0. Hence $\text{im } \rho' = \text{im } \rho$. $\square$

Part (iv) is the delicate one when $\text{im } \rho = 0$: there the leftover gluing is the only place a rotation could enter, and it does not.

So one loop of gap γ realizes the split $(\gamma + 2, w + 2 - \gamma)$, and as γ runs over $1, \dots, w - 1$ the smaller part runs over $3, \dots, w + 1$. The extreme split producing valence 1 comes from varying the surgery rather than the loop, and it behaves differently enough to deserve its own statement.

Lemma 4.2 (Odd twist). *In the situation of Lemma 4.1, set instead*

$$\alpha'(x) = y_1, \quad \alpha'(\alpha(x)) = y_0, \quad \alpha'(y_2) = y_3,$$

so that the two halves of the cut edge go to adjacent sides of the new square and the leftover sides y_2, y_3 , also adjacent, are glued to each other. Then M' is a closed quad mesh with one

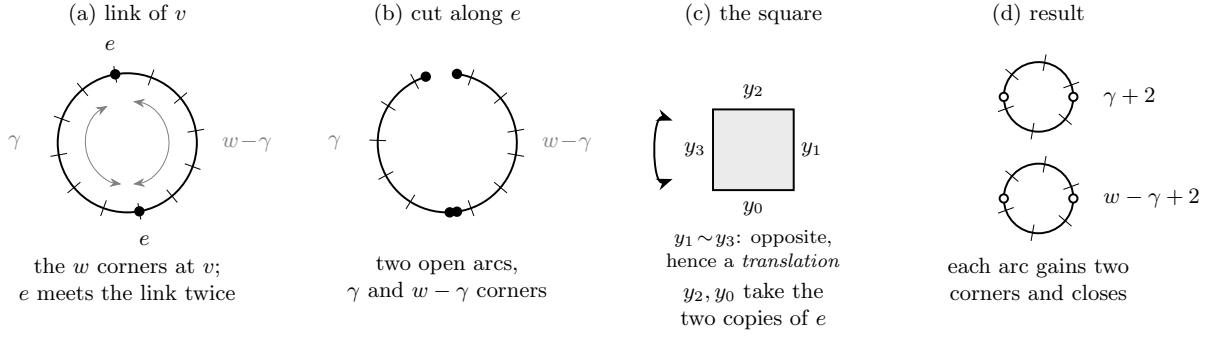


Figure 5: The splitting surgery of Lemma 4.1, seen in the link of the cone. (a) The w corners at v form a circle, which the loop e meets twice, separating γ corners from $w - \gamma$. (b) Cutting along e opens the circle into two arcs. (c) One square is glued in. Its *opposite* sides y_2, y_0 take the two copies of e , which is the half-turn, and its two remaining sides y_1, y_3 , also opposite, are glued to each other. The rotation of that last gluing is $(3 - 1 + 2) \equiv 0$, a translation; by a half-turn it would be 2, which would enter $\text{im } \rho$ and break part (iv). (d) Each arc receives two of the square's four corners, drawn hollow, and closes up, giving cones of valence $\gamma + 2$ and $w - \gamma + 2$.

more square, the same genus, every other vertex untouched, and v replaced by two vertices of valences

$$1 \quad \text{and} \quad w + 3,$$

independently of the gap γ . Its holonomy is $\text{im } \rho' = \mathbb{Z}_4$, whatever $\text{im } \rho$ was.

Proof. Cutting is as before. For the filling, the square's four corners are $c_0, \dots, c_3$ with c_t between sides y_{t-1} and y_t . Gluing y_2 to y_3 folds the corner c_3 between them onto itself, so c_3 becomes a vertex with exactly one corner: valence 1. The same identification joins the far end of one slit arc to the far end of the other, so the two arcs, carrying γ and $w - \gamma$ corners, are no longer separated; together with the three remaining corners c_0, c_1, c_2 they form a single vertex of $\gamma + (w - \gamma) + 3 = w + 3$ corners. This is why γ drops out, and Euler's count is as before: F and V rise by one, E by two.

For the holonomy, the rotation across the leftover edge is $(3 - 2 + 2) \equiv 3 \pmod{4}$, an odd number, because y_2 and y_3 are *adjacent* rather than opposite. That edge is dual to a loop of the dual graph, so $3 \in \text{im } \rho'$, and a subgroup of $\mathbb{Z}_4$ containing an odd element is $\mathbb{Z}_4$. $\square$

So the odd twist is available exactly when $d = 1$, and that is exactly when it is needed, since valence 1 means order -3 , which is odd, and $d \mid m_i$ then forces $d = 1$. The constraint is automatic, not a side condition to check.

For valence 2 the algorithm uses a third move: cut *two* edges and rejoin the eight loose sides by one of the 105 pairings. We claim no lemma for it. It is not a subdivision, since the old dual cycles need not survive, so it can lower $\text{im } \rho$ as well as raise it, and different pairings give different valence multisets. Its outputs are admitted only when Theorem 4.6 applies to them, which is why the construction needs no general statement about the move itself.

4.2 Base meshes

Lemma 4.3 (Base meshes). *For every $g \geq 2$ the following gluing of $N = 2g - 1$ unit squares is a closed quad mesh of genus g with a single vertex, of valence $8g - 4$, and $\text{im } \rho = \mathbb{Z}_4$. Numbering*

the darts $0, \dots, 4N - 1$ so that dart $4f + s$ is side s of square f , as in Figure 6 for $g = 2$, glue

$$\begin{array}{ll} (0, 2), (1, 4), (3, 5) & \text{first cap} \\ (s, s + 2), (s + 1, s + 4), (s + 3, s + 6), (s + 5, s + 7) & \text{blocks, } s = 6 + 8b, b = 0, \dots, g - 3 \\ (t, t + 2), (t + 1, t + 4), (t + 3, t + 5) & \text{last cap, } t = 4N - 6. \end{array}$$

Replacing this by the staircase pattern

$$(0, 2), \quad (2i - 1, 2i + 2) \text{ for } i = 1, \dots, 2N - 2, \quad (4N - 3, 4N - 1)$$

gives, for the same N , a one-vertex mesh of genus g with trivial holonomy: a translation surface in the minimal abelian stratum $\mathcal{H}(2g - 2)$.

Proof. Each pattern uses every dart exactly once, so it defines a closed mesh, with $2N = 4g - 2$ edges, matching $3 + 4(g - 2) + 3$ in the first case.

One vertex. This says $\nu = \alpha \circ \sigma^{-1}$ is a single $4N$ -cycle; we induct on g . For $g = 2$, ν runs $0, 5, 1, 2, 4, 10, 11, 7, 8, 9, 6, 3$ and closes: one 12-cycle.

For the step put $t = 4N - 6$. Below $4N$ the genus- $(g + 1)$ pattern differs from the genus- g one in a single pair: the cap's $(t + 3, t + 5)$ becomes $(t + 3, t + 6)$ and $(t + 5, t + 7)$, while $(t, t + 2)$ and $(t + 1, t + 4)$ survive as the first two pairs of the new block. Since $\sigma^{-1}(t + 4) = t + 3$ and $\sigma^{-1}(t + 2) = t + 5$, and no other dart maps into $\{t + 3, t + 5\}$, exactly two values of ν change,

$$\nu(t + 4) : t + 5 \mapsto t + 6, \quad \nu(t + 2) : t + 3 \mapsto t + 7,$$

and on the eight new darts the new block and cap give

$$t + 6 \mapsto t + 12 \mapsto t + 13 \mapsto t + 9 \mapsto t + 10 \mapsto t + 11 \mapsto t + 8 \mapsto t + 5, \quad t + 7 \mapsto t + 3.$$

So the old cycle's step $t + 4 \rightarrow t + 5$ is replaced by an arc through seven new darts and its step $t + 2 \rightarrow t + 3$ by the single new dart $t + 7$. Replacing two steps of a cycle by two disjoint arcs covering all eight new darts leaves one cycle, of length $4N + 8$.

The staircase is the same argument: with $M = 4N$, the two changed values are at $M - 4$ and $M - 2$, and the steps $M - 4 \rightarrow M - 3$ and $M - 2 \rightarrow M - 1$ become $M - 4 \rightarrow M + 2 \rightarrow M + 4 \rightarrow M + 5 \rightarrow M + 1 \rightarrow M - 3$ and $M - 2 \rightarrow M \rightarrow M + 6 \rightarrow M + 7 \rightarrow M + 3 \rightarrow M - 1$.

Euler's formula then gives $\chi = 1 - 2N + N = 1 - N = 2 - 2g$.

The holonomy is not forced by the cone, whose order $8g - 8$ is divisible by 4. Both claims about $\text{im } \rho$ have closed-form proofs, valid for every g .

For the staircase, assign to square f the chart rotation $r_f = f \bmod 4$; we check that r trivialises every transition, that is that $r_h - r_f$ equals the rotation across each glued pair, which forces $\text{im } \rho = 0$. The pair $(0, 2)$ joins sides 0 and 2 of square 0 to itself: rotation $(2 - 0 + 2) \equiv 0$, and $r_0 - r_0 = 0$. For a middle pair $(2i - 1, 2i + 2)$, write $i = 2j$ or $i = 2j + 1$. If $i = 2j$ the darts are $4(j - 1) + 3$ and $4j + 2$, so the rotation is $(2 - 3 + 2) \equiv 1$ and $r_j - r_{j-1} = 1$. If $i = 2j + 1$ they are $4j + 1$ and $4(j + 1) + 0$, so the rotation is $(0 - 1 + 2) \equiv 1$ and $r_{j+1} - r_j = 1$. The last pair $(4N - 3, 4N - 1)$ joins sides 1 and 3 of square $N - 1$: rotation $(3 - 1 + 2) \equiv 0$, and the difference is 0. Every edge is accounted for, so ρ is the coboundary of r and vanishes on every cycle.

For the first pattern it suffices to exhibit one dual cycle of odd rotation, since a subgroup of $\mathbb{Z}_4$ containing an odd element is $\mathbb{Z}_4$. The first cap glues $(1, 4)$ and $(3, 5)$; darts 1, 3 lie in square 0 and darts 4, 5 in square 1, so these two edges form a cycle of the dual graph. Its rotation is $(0 - 1 + 2) + (3 - 1 + 2) \equiv 1 + 0 = 1$, odd. The first cap is present for every $g \geq 2$, so $\text{im } \rho = \mathbb{Z}_4$ throughout. $\square$

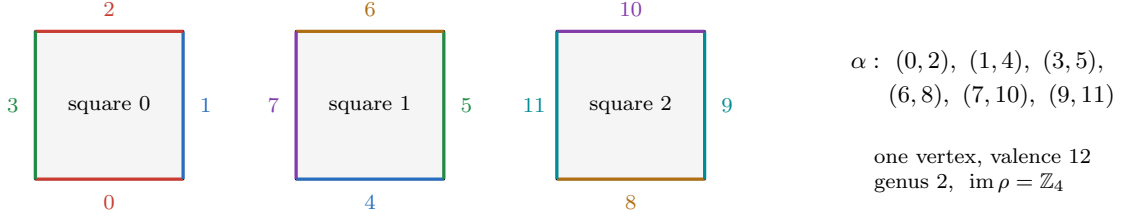


Figure 6: The base mesh of Lemma 4.3 at $g = 2$: $N = 2g - 1 = 3$ unit squares, side s of square f carrying dart $4f + s$, numbered counterclockwise from the bottom. The six gluings, one colour each, pair every dart once, so the mesh is closed. Here $\nu = \alpha \circ \sigma^{-1}$ is a single 12-cycle, so there is exactly one vertex, of valence $8g - 4 = 12$, a cone of angle 6π and order 8. Euler's formula gives $\chi = 1 - 6 + 3 = -2$, so $g = 2$. The edges $(1, 4)$ and $(3, 5)$ both join squares 0 and 1 and form a dual cycle of rotation $1 + 0 = 1$, which is odd and forces $\text{im } \rho = \mathbb{Z}_4$. This pair is the first cap of the general pattern and is present for every $g \geq 2$.

Remark 4.4 (the parity constraint). Searching for base meshes with $\text{im } \rho \subseteq 2\mathbb{Z}_4$ is far easier than it looks, because of a structural observation. The rotation across an edge joining side s to side t is $(t - s + 2) \bmod 4$, which is even exactly when $s \equiv t \pmod{2}$. So if every gluing joins sides of equal parity, every rotation is even and hence $\text{im } \rho \subseteq 2\mathbb{Z}_4$. Conversely, if $\text{im } \rho \subseteq 2\mathbb{Z}_4$ then the parities of the rotations form a $\mathbb{Z}/2$ -coboundary, so after rotating some charts by a quarter turn, which changes ρ by a coboundary and not $\text{im } \rho$, every gluing does join sides of equal parity. Restricting a search to parity-preserving involutions therefore loses nothing and cuts the space in half twice over; it is how the $d = 2$ bases below were found. Note also that σ reverses parity while α preserves it, so ν reverses it and every vertex of such a mesh has even valence, as it must, since $2 \mid m_i$.

4.3 The construction and what its output certifies

Algorithm 4.5. *Let $g \geq 2$, $d \in \{1, 2, 4\}$, and let $m_1, \dots, m_n$ be integers with $m_i > -4$, $d \mid m_i$ and $\sum_i m_i = 4(2g - 2)$.*

1. *Start from a base mesh with $\text{im } \rho = d\mathbb{Z}_4$: Lemma 4.3 for $d = 1$ and $d = 4$, for every $g \geq 2$. For $d = 2$ we have no closed-form family; base meshes are recorded for $2 \leq g \leq 8$, found by search under Remark 4.4, so the construction as it stands runs in those genera when $d = 2$. For $3 \leq g \leq 8$ the recorded base is the minimal one, with $2g - 1$ squares and a single cone of valence $8g - 4$, the stratum $Q(4g - 4)$; only $g = 2$ needs two cones, for the reason of Remark 4.8: $Q(4)$ is empty, so $Q(2, 2)$ is used.*
2. *Repeatedly pick a cone of the current mesh and split it by one square-insertion surgery (Lemma 4.1, Lemma 4.2, or the figure-eight move), so that one of the two pieces is a cone the target asks for.*
3. *Backtrack over which cone to split, which target valence to peel off it, and which surgery to use.*

Theorem 4.6 (Certificate theorem). *Let A be any gluing array on $2g - 2 + n$ squares, in particular one output by Algorithm 4.5. Suppose that, read off from A alone, it defines a closed connected mesh whose vertex valences are $m_i + 4$ and whose dual holonomy has image $d\mathbb{Z}_4$. Then*

A is a seamless parametrization with signature (m, ρ) for some ρ with $\text{im } \rho = d\mathbb{Z}_4$; equivalently, a 4-differential η^d with η primitive of order $4/d$. Its stratum is therefore non-empty, and $2g - 2 + n$ is the smallest number of squares the corner count allows.

Proof. The first statement is Remark 2.2: a closed quad mesh is a seamless parametrization, a vertex of valence v is a cone of order $v - 4$, and $\text{im } \rho$ is what the dual holonomy computes, all of which is determined by A alone.

For minimality, let M' be *any* closed quad mesh realizing the signature. Besides the n prescribed cones it may carry $r \geq 0$ further vertices, which are then regular, of valence 4. Every corner belongs to exactly one vertex, so by Gauss–Bonnet

$$4F = \sum_i (m_i + 4) + 4r = 4(2g - 2) + 4n + 4r,$$

whence $F = 2g - 2 + n + r \geq 2g - 2 + n$. Such an A has exactly $2g - 2 + n$ squares, so it attains the bound. $\square$

The point of stating it this way is that the conclusion depends on the *output*, not on the moves that produced it. Nothing is assumed about which surgeries the algorithm used, including the figure-eight move, for which no valence formula is claimed, because every invariant is re-derived from the final dart pairing.

Remark 4.7 (what is and is not proved here). Whenever the algorithm returns, it returns an explicit gluing, and Theorem 4.6 turns that gluing into a complete proof for its signature, resting on nothing unproved. Three things are *not* proved. Step 2 may in principle get stuck; ruling that out needs a statement of the form “a cone of valence w on a genus ≥ 2 mesh always carries a loop of the required gap, possibly after remeshing”, which the backtracking search sidesteps in practice but which we have not established. The figure-eight move, the only local move without a general lemma, has no valence formula, proved or conjectured; its outputs are admitted only because their invariants are verified directly. And for $d = 2$ base meshes are available only for $2 \leq g \leq 8$, by search rather than in closed form. What the section establishes is therefore an unconditional correctness statement for whatever the algorithm outputs, together with non-emptiness for the data it has been run on, not a universally quantified existence theorem.

Since splitting only ever raises the number of cones, targets with no more cones than the base are out of reach by surgery; those are handled instead by an exhaustive search over all gluings of the minimal size. A positive result there is a minimal witness; a negative one rules out only realizations of minimal size, since a larger mesh with extra regular vertices could still exist. Global non-realizability comes from Theorem 3.10 and (S7), never from a bounded search.

Remark 4.8 (a sharp check). Run over the genus-two quadratic strata, that is $d = 2$ and $g = 2$, the construction builds every stratum except $Q(4)$ and $Q(1, 3)$, on which it fails. These are exactly the two that [13] classifies as empty. In genus 2 the base mesh for $d = 2$ cannot be the minimal stratum for the same reason: $Q(4)$ is empty, so $Q(2, 2)$ is used instead.

5 Signatures with non-coprime cone orders

Theorem 3.10 answers every admissible signature, so in particular it answers the ones no earlier criterion could. This section pins down that region and works through it.

5.1 Where the gcd condition stops

The hypothesis of [14, Prop. 2] is $\gcd(I_1, \dots, I_n) = \frac{1}{4}$ on the cone indices, the loop holonomies not entering, and with $I_i = -m_i/4$ this reads

$$\gcd_{\mathbb{Z}}(m_1, \dots, m_n) = 1.$$

It is what makes their rerouting able to hit prescribed turning numbers exactly, and under it every signature is realizable but the torus with cones $3\pi/2, 5\pi/2$. Three consequences fix the region it leaves open.

It fails whenever every cone angle is a multiple of π , since then every m_i is even. This is the case [14] itself singles out, “indices restricted to multiples of $\frac{1}{2}$ ”, called there a realistic scenario, and it is where families 3, 4 and 5 live.

It fails whenever there is only one cone, since then $\gcd_{\mathbb{Z}}(m) = |m_1| = 4(2g - 2)$, which is 1 for no g . A single-cone signature is never covered, whatever its angle.

It is strictly stronger than $\text{im } \rho = \mathbb{Z}_4$. One odd m_i gives $\langle m_i \rangle = \mathbb{Z}_4$ and so $\text{im } \rho = \mathbb{Z}_4$, but not conversely: at $g = 4$ the angles $7\pi/2$ and $25\pi/2$ give $m = (3, 21)$ with $\gcd_{\mathbb{Z}} = 3$. The difference is exactly the one noted after Corollary 3.5, turning numbers in $\frac{1}{4}\mathbb{Z}$ against a holonomy class in $\mathbb{Z}_4$, and it means the uncovered region is larger than the middle and right columns of Figure 1: it is all of $\{\gcd_{\mathbb{Z}}(m_i) \neq 1\}$.

That region is not small. A cross field made of two globally distinguishable line fields has $\text{im } \rho \subseteq 2\mathbb{Z}_4$ automatically, since transport around a loop preserves the labelling of the two families and so turns the cross by 0 or by π ; then every m_i is even and the condition cannot fire. Fields of that kind are ordinary: the principal directions away from the umbilics, where $\kappa_1 > \kappa_2$ singles out one of the two lines; anything built from a quadratic differential; any stripe or ridge-valley pattern. A singularity of index i appears there as a cone of angle $2\pi(1 - i)$, so umbilics of index $+\frac{1}{2}$ and $-\frac{1}{2}$ are cones of angle π and 3π . What [14] observes is a different and compatible thing: a cross field *optimized* for smoothness or curvature alignment is a genuine 4-symmetry field and almost always acquires an index- $\pm\frac{1}{4}$ singularity, which restores the condition.

5.2 What the classification says there

Corollary 5.1. *Let $s = (g, m, \rho)$ be a Gauss–Bonnet-admissible signature whose cone orders are not coprime, that is $\gcd_{\mathbb{Z}}(m_1, \dots, m_n) \neq 1$, so that [14, Prop. 2] does not apply. Then s is realizable except for exactly four signatures:*

$$\begin{aligned} g = 1, \text{ no cones, } \text{im } \rho \neq 0; & \quad g = 1, \text{ angles } \pi, 3\pi, \text{ im } \rho = 2\mathbb{Z}_4; \\ g = 2, \text{ angle } 6\pi, \text{ im } \rho = 2\mathbb{Z}_4; & \quad g = 2, \text{ angles } 3\pi, 5\pi, \text{ im } \rho = 2\mathbb{Z}_4. \end{aligned}$$

In particular, on that region: if some m_i is odd, s is realizable; and if $\text{im } \rho = 0$, that is if the cross field lifts to a global vector field, s is realizable, in every genus.

Proof. By Theorem 3.10 the unrealizable signatures are the five families of Table 1. Family 2 has $m = (1, -1)$, so $\gcd_{\mathbb{Z}} = 1$ and it is excluded by hypothesis; it is the exception [14] already records. The other four have $m = \emptyset, (2, -2), (8)$ and $(6, 2)$, of $\gcd_{\mathbb{Z}}$ equal to 0, 2, 8 and 2, so all four lie in the region and are the only failures there.

For the two special cases: none of the four has an odd m_i , which gives the first; and $\text{im } \rho = 0$ forces $4 \mid m_i$, which none of the four satisfies. The second also follows directly from Theorem 3.6, since $d = 4$ makes $k = 1$, and $m_i \geq 0$ then rules out all three cases of (S7), so no abelian stratum in play is empty. $\square$

genus	cone angles	$\gcd_{\mathbb{Z}}(m_i)$	$\text{im } \rho$		
			$\mathbb{Z}_4$	$2\mathbb{Z}_4$	0
1	none	0	$\times (1)$	$\times (1)$	$\checkmark 1$
1	$\pi, 3\pi$	2	$\checkmark 2$	$\times (3)$	n/a
1	$\pi, \pi, 3\pi, 3\pi$	2	$\checkmark 4$	$\checkmark 4$	n/a
2	6π	8	$\checkmark 3$	$\times (4)$	$\checkmark 3$
2	$3\pi, 5\pi$	2	$\checkmark 4$	$\times (5)$	n/a
2	$3\pi, 3\pi, 3\pi, 3\pi$	2	$\checkmark 6$	$\checkmark 6$	n/a
3	10π	16	$\checkmark 5$	$\checkmark 5$	$\checkmark 5$
4	$7\pi/2, 25\pi/2$	3	$\checkmark 8$	n/a	n/a

Table 2: Eight signatures against the three possible holonomies. *No row has $\gcd_{\mathbb{Z}}(m_i) = 1$, so none of them is covered by [14, Prop. 2].* “ $\checkmark N$ ” means realizable, N being the smallest number of squares any realization can have (Theorem 4.6); “ $\times (k)$ ” means unrealizable, k being the family of Table 1 responsible; “n/a” means the holonomy is inadmissible for those angles, since d must divide every m_i . The rows with mixed verdicts are the point: the surface and the cone angles are fixed, and $\text{im } \rho$ alone decides. The last row has an odd cone order, so $\text{im } \rho = \mathbb{Z}_4$ is forced (Corollary 3.5) and the answer is easy, but the gcd is 3, not 1.

So four of the five unrealizable families lie outside the gcd condition, and the fifth is the exception it already records. Everything genuinely new in Theorem 3.10 is in the region this corollary describes.

5.3 Examples

1. *A torus with two umbilics.* Cones of angles π and 3π , in the curvature-aligned reading umbilics of index $+\frac{1}{2}$ and $-\frac{1}{2}$. With $\text{im } \rho = \mathbb{Z}_4$ there is a realization, and the smallest has two squares: Figure 7(a). With $\text{im } \rho = 2\mathbb{Z}_4$ there is none, at any size. The reduced stratum is $Q(1, -1)$ on the torus, empty by Proposition 3.8, or by (S7)(i) with $k = 2$, and this is family 3. So a torus whose principal directions have exactly two umbilics, one of each sign, admits no quadrangulation aligned to them. It is the exact analogue, one rung down, of the $3\pi/2$ – $5\pi/2$ torus of [14, 7]; it is not that exception, since here every angle is a multiple of π , so the gcd condition does not apply.

2. *Two more cones remove the obstruction.* Keep the torus and $\text{im } \rho = 2\mathbb{Z}_4$, but ask for angles $\pi, \pi, 3\pi, 3\pi$, two umbilics of each sign. Now there is a realization, and Figure 7(b) is one, on the four squares that Theorem 4.6 shows to be the minimum. So the obstruction of Example 1 is not about the torus, and not about the angles: it is about that pair of angles together with that holonomy. This is the kind of statement a classification can make and a sufficient condition cannot.

3. *One cone, three answers.* Genus 2 with a single cone of angle 6π , the minimal case, a cone of valence 12. For $\text{im } \rho = \mathbb{Z}_4$ the base mesh of Lemma 4.3 realizes it on $2g - 1 = 3$ squares, drawn at $g = 2$ in Figure 6. For $\text{im } \rho = 0$ the staircase of the same lemma realizes it on three squares, as a translation surface in $\mathcal{H}(2)$. For $\text{im } \rho = 2\mathbb{Z}_4$ it is not realizable at all: the reduced stratum is $Q(4)$, which is empty, and this is family 4. Same surface, same singularity, three cross fields, and the answers are yes, no, yes. The obstruction shows up already at minimal size,

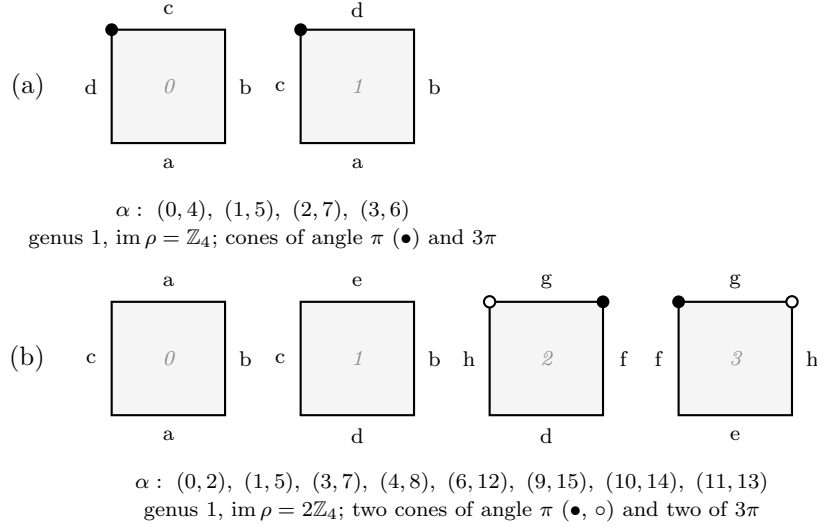


Figure 7: Two explicit minimal witnesses on the torus, in the notation of Figure 6: side s of square f carries dart $4f + s$, numbered counterclockwise from the bottom, and edges bearing the same letter are glued. Marked corners are the cones of angle π , that is the vertices of valence 2, where only two squares meet; the remaining corners fall into the cones of angle 3π . **(a)** The signature of Example 1 with $\text{im } \rho = \mathbb{Z}_4$. Change the holonomy to $2\mathbb{Z}_4$ and no such mesh exists at any size, by family 3. **(b)** Adding one cone of each angle restores realizability with $\text{im } \rho = 2\mathbb{Z}_4$: the reduced stratum becomes $Q(1, 1, -1, -1)$, which is non-empty, and here is a mesh in it.

since among all gluings of three squares those of genus 2 with a single cone of valence 12 realize $\text{im } \rho = \mathbb{Z}_4$ and $\text{im } \rho = 0$ and never $2\mathbb{Z}_4$; but what rules it out at *every* size is Theorem 3.10.

4. *Genus three: the obstruction is gone.* A single cone of angle 10π on a genus-3 surface is realizable for all three holonomies, each on the minimal $2g - 1 = 5$ squares: Lemma 4.3 for $\text{im } \rho = \mathbb{Z}_4$ and for $\text{im } \rho = 0$, and the recorded base mesh for $\text{im } \rho = 2\mathbb{Z}_4$, which is the minimal quadratic stratum $Q(8)$. Families 4 and 5 are genus-two phenomena and not the first terms of a pattern: by (S7) no primitive quadratic stratum in genus ≥ 3 is empty.

5. *And the genus-two pair.* Cones of angles 3π and 5π with $\text{im } \rho = 2\mathbb{Z}_4$ is family 5, reduced stratum $Q(1, 3)$: a genus-two surface whose principal directions have exactly two umbilics, of index $-\frac{1}{2}$ and $-\frac{3}{2}$, admits no aligned quadrangulation. Raise the holonomy to $\text{im } \rho = \mathbb{Z}_4$ and four squares suffice. Keep $\text{im } \rho = 2\mathbb{Z}_4$ but spread the same total curvature over four cones of angle 3π , four umbilics of index $-\frac{1}{2}$, and six squares suffice.

6. *An odd cone that the condition still misses.* At $g = 4$, cones of angles $7\pi/2$ and $25\pi/2$: here $m = (3, 21)$, so $\gcd_{\mathbb{Z}} = 3$ and [14, Prop. 2] does not apply, while 3 is odd, so Corollary 3.5 forces $\text{im } \rho = \mathbb{Z}_4$ and leaves a single orbit. Corollary 5.1 says realizable, and Algorithm 4.5 returns a mesh on the minimal $2g - 2 + n = 8$ squares. This is the easy part of the uncovered region, with no coprimality, no cone of angle $3\pi/2$ or $5\pi/2$, and still nothing to check, and it is uncovered only because turning numbers in $\frac{1}{4}\mathbb{Z}$ are a finer thing to prescribe than a holonomy class in $\mathbb{Z}_4$.

6 Extensions and open problems

6.1 Surfaces with boundary: feature curves

Cutting a surface along a network of feature curves produces a compact surface with boundary on which the parametrization must send the boundary to axis-parallel segments. This subsection carries the theory over. Let M be compact oriented of genus g with $b \geq 1$ boundary components and interior marked points C . Boundary vertices are corners of angle $\alpha = a\pi/2$ with $a \geq 1$; a corner with $a = 2$ is a straight point and carries no information. The signature records the interior orders m_i , the cyclic sequences of corner angles, and ρ , with

$$\rho(\partial_j) \equiv t_j := \sum_k (2 - a_{j,k}) \pmod{4},$$

the *turning* of the j -th boundary component in quarter turns; this is forced, since the boundary is geodesic between corners.

Lemma 6.1 (Gauss–Bonnet with corners). *For a feature-aligned structure, $\sum_i (4 - v_i) + \sum_{j,k} (2 - a_{j,k}) = 4\chi(M) = 4(2 - 2g - b)$, where $v_i = m_i + 4$. Consequently $\sum_i m_i + \sum_{j,k} a_{j,k}$ is even.*

Proof. Gauss–Bonnet for a flat metric with conical singularities and geodesic boundary with corners reads $\sum_i (2\pi - \theta_i) + \sum_{j,k} (\pi - \alpha_{j,k}) = 2\pi\chi$; divide by $\pi/2$. Reducing modulo 2 gives the parity statement. $\square$

The parity constraint is necessary but not sufficient for quad-mesh realizability. The total corner count is $4F = \sum_i v_i + \sum_{j,k} a_{j,k} = 4n + \sum_i m_i + \sum_{j,k} a_{j,k}$, so divisibility by four asks for $\sum_i m_i + \sum_{j,k} a_{j,k} \equiv 0 \pmod{4}$, whereas Lemma 6.1 gives only $\equiv 0 \pmod{2}$; the remaining factor of two is the evenness of the boundary edge count, forced separately by $4F = 2E_{\text{int}} + E_{\partial}$. The two conditions are independent.

Lemma 6.2. *$H_1(M \setminus C; \mathbb{Z})$ is free of rank $2g + b + n - 1$; the classes γ_i and ∂_j satisfy the single relation $\sum_i \gamma_i = \sum_j \partial_j$; and $H_1(M \setminus C)/\langle \gamma_i, \partial_j \rangle \cong H_1(M)/\langle \partial_j \rangle \cong \mathbb{Z}^{2g}$ with unimodular induced intersection form. Hence the set of signatures with prescribed orders and turnings is a coset of $H^1(M; \mathbb{Z}_4) \cong \mathbb{Z}_4^{2g}$.*

Proof. The first two statements are standard. Capping each boundary component with a disk gives a closed genus- g surface and identifies the quotient with its first homology, on which the intersection form is the standard symplectic one. Two signatures with the same orders and turnings differ by a homomorphism killing all γ_i and ∂_j , that is by an element of $\text{Hom}(\mathbb{Z}^{2g}, \mathbb{Z}_4)$. $\square$

Theorem 6.3 (Reduction Lemma with boundary). *Let $g \geq 1$ and fix the interior orders and the corner data, and put $D = \langle m_1, \dots, m_n, t_1, \dots, t_b \rangle \leq \mathbb{Z}_4$. Two feature-aligned signatures with these data lie in the same orbit of the group of homeomorphisms preserving C and each boundary component with its corner sequence if and only if they have the same $\text{im } \rho$; the number of orbits is the number of subgroups of $\mathbb{Z}_4$ containing D .*

Proof. As for Theorem 3.4, with the three ingredients adapted. Lemma 3.1 applies unchanged to pushing an interior cone, and also to *sliding a boundary component*: a collar of ∂_j plays the role of the puncture, the slide along α is again a product $T_{\alpha_L} T_{\alpha_R}^{-1}$ of twists along the boundary curves of an annulus containing the collar, and the same computation gives $x \mapsto x + \langle x, \alpha \rangle [\partial_j]$, hence

$\rho \mapsto \rho + t_j \langle \cdot, \alpha \rangle$; the slide preserves the corner sequence. By Lemma 6.2 these moves give exactly the translations by $D \cdot \mathbb{Z}_4^{2g}$. Lemma 3.2 is unchanged. In Lemma 3.3, replace the disk D_0 by an embedded genus-0 subsurface P containing C and all b boundary components of M and having one further boundary circle; then $\Sigma = M \setminus \text{int}(P)$ has genus g and one boundary circle, and setting $\rho_0 = 0$ on a symplectic basis inside Σ is consistent because $\rho_0([\partial\Sigma]) = \sum_i m_i - \sum_j t_j = 0$ by Lemma 6.2. The rest of the argument is verbatim. $\square$

Corollary 6.4. *If some interior order m_i is odd, or some boundary component has odd turning t_j , equivalently an odd number of corners of odd angle, then $D = \mathbb{Z}_4$, a single orbit remains, and realizability does not depend on ρ .*

A single right-angle corner does *not* by itself make t_j odd: a component with one corner of angle $\pi/2$ and one of angle $3\pi/2$ has $t_j = (2-1) + (2-3) = 0$. What is needed is an odd *number* of odd-angle corners on the same component. Feature networks in practice are full of odd-angle corners, and nothing makes their number even on every component, so $D = \mathbb{Z}_4$ is the normal state of affairs and the rotations along homology loops are usually not part of the problem. This is a precise reason why the sufficient conditions in the feature-aligned literature work as well as they do.

Lemma 6.5 (Doubling). *Let $\widetilde{M} = M \cup_{\partial} \overline{M}$ be the double, a closed surface of genus $2g + b - 1$. A feature-aligned structure on M doubles to a flat $\mathbb{Z}_4$ -cone structure on $\widetilde{M}$ in which each interior cone of order m_i gives two cones of order m_i , each boundary corner of angle $a\pi/2$ gives one interior cone of angle $a\pi$, that is of order $2a - 4$, and $\text{im } \tilde{\rho} \supseteq \langle \text{im } \rho, 2a \bmod 4 : \text{all corners} \rangle$.*

Proof. Reflecting the atlas across each boundary arc extends it over the mirror copy: a boundary arc develops to an axis-parallel segment, and reflection in such a segment is an isometry of the plane, so after re-orienting the mirror charts the doubled atlas again has transitions in $\mathbb{Z}_4 \times \mathbb{R}^2$. The genus is $2g + b - 1$ by Euler characteristic. A boundary corner of angle α becomes an interior point of total angle 2α , and an interior cone is duplicated. Finally $\text{im } \tilde{\rho}$ contains $\text{im } \rho$ because $H_1(M)$ maps to $H_1(\widetilde{M})$, and it contains the loops around the new cones, whose $\tilde{\rho}$ -values are $2a - 4 \equiv 2a$. $\square$

The inclusion can be strict: crossing the boundary contributes a rotation by π whenever the two reflected charts differ in direction, so the double can carry holonomy the signature of M does not. The smallest example is a genus-0 surface with three straight boundary circles and trivial ρ , whose double has $\text{im } \tilde{\rho} = 2\mathbb{Z}_4$.

Corollary 6.6. *If every closed signature consisting of genus $2g + b - 1$, the doubled orders of Lemma 6.5, and a subgroup $S \supseteq \langle \text{im } \rho, 2a \bmod 4 \rangle$ is unrealizable, then the feature-aligned signature is unrealizable.*

This is the only direction doubling gives for free, and by itself it gives nothing. Over all 3058 admissible feature-aligned signatures whose corner count fits in four squares, Corollary 6.6 never applies. The reason is structural: the doubled orders are even at the corners, so a double that could land in one of the empty quadratic strata has $\text{im } \tilde{\rho} = 2\mathbb{Z}_4$, while the same orders with $\text{im } \tilde{\rho} = \mathbb{Z}_4$ give a primitive 4-differential, and those strata are non-empty. Deciding the boundary case therefore requires non-emptiness of *real* strata, meaning k -differentials invariant under a prescribed anti-holomorphic involution with the feature network as fixed locus, which is the main question this paper leaves open on that side.

6.2 Fixed conformal structure

A second, differently quantified answer exists in the literature: at a *fixed* conformal structure and *fixed* cone positions there is an Abel–Jacobi criterion [11, 12]. That line of work replaced the earlier quadratic-differential (foliation) approach of [10] by meromorphic *quartic* differentials precisely in order to reach singularities of odd topological valence, and Corollary 6.9 below is the reason that step was necessary. The following makes the relation to the present paper precise.

Proposition 6.7. *Let X be a Riemann surface of genus g , $p_1, \dots, p_n$ distinct points, and $m_i > -4$ integers with $\sum_i m_i = 4(2g - 2)$. Put*

$$E(X, p, m) = \left\{ e \mid 4 : e \mid m_i \ \forall i \text{ and } \sum_i (m_i/e)p_i \sim (4/e)K_X \right\},$$

where $\sim$ denotes linear equivalence. Then a 4-differential q with $\text{div}(q) = \sum_i m_i p_i$ exists if and only if $1 \in E$, in which case q is unique up to scale and $\text{im } \rho_q = d\mathbb{Z}_4$ with $d = \max E(X, p, m)$.

Proof. A meromorphic section of K_X^4 with prescribed divisor exists if and only if the bundle

$$K_X^4 \otimes \mathcal{O}(-\sum_i m_i p_i)$$

is trivial, which is the displayed linear equivalence for $e = 1$, and it is then unique up to a constant. By Lemma 2.6 we have $q = \eta^e$ for a $(4/e)$ -differential η exactly when $\text{im } \rho_q \subseteq e\mathbb{Z}_4$, and such an η has $\text{div}(\eta) = \sum (m_i/e)p_i$, so it exists if and only if $e \in E$. (Given $e \in E$ one gets η with $\eta^e = cq$ for some constant $c \neq 0$; replacing η by $c^{-1/e}\eta$ makes $\eta^e = q$ exactly.) Since the subgroups of $\mathbb{Z}_4$ are totally ordered, $\text{im } \rho_q$ corresponds to the largest such e . $\square$

Thus at fixed (X, p) the holonomy is not free but computed. On a torus, where K is trivial, $e \in E$ reads $\sum_i (m_i/e)c_i = 0$ in the group law, which is exactly conditions (2) and (3) of Proposition 3.8.

Corollary 6.8. *A holonomy signature with orders m and $\text{im } \rho = d\mathbb{Z}_4$ is realizable if and only if there exist a Riemann surface X of genus g and distinct points p_i with $\max E(X, p, m) = d$.*

So the fixed-structure criterion is the pointwise condition, and the question of this paper is whether the locus it cuts out is non-empty. That is what the parametrization problem needs, a parametrization being free to choose its flat metric.

Corollary 6.9. *If some cone has odd topological valence, that is if some m_i is odd, then $E(X, p, m) \subseteq \{1\}$ for every (X, p) . Hence if a 4-differential with divisor $\sum_i m_i p_i$ exists on (X, p) at all, it is primitive, with $\text{im } \rho = \mathbb{Z}_4$.*

Proof. $e \in E$ requires $e \mid m_i$ for all i ; an odd m_i forces $e = 1$. $\square$

The inclusion cannot be strengthened to an equality. Whether $1 \in E$, that is whether q exists at all on that particular (X, p) , is the Abel–Jacobi condition of Proposition 6.7, and it fails for most (X, p) . An odd order rules out a nontrivial power; it does not produce a differential.

Odd valence is therefore exactly the regime $D = \mathbb{Z}_4$ of Corollary 3.5: the single orbit, the case in which the rotations along homology loops carry no information, and the case in which the object is a genuine 4-differential rather than a square or a fourth power. A method built on quadratic differentials sees only $e \geq 2$ and cannot reach it; for the topological question the same regime is the easiest one.

6.3 Open problems

- (1) *The external input.* Theorem 3.10 rests on (S7) and on nothing else outside §§2–3. Its genus-one cases are reproved here as Proposition 3.8; a self-contained proof of the genus-two case $k = 2$, and of the assertion that the list stops there, would remove the dependence on (S7) and make the higher-genus non-emptiness argument self-contained.
- (2) *Completeness of the construction.* Three gaps, in increasing order of difficulty. First, prove a general lemma for the figure-eight move, including its vertex, genus and holonomy formulas. Cones of order -2 come only from that move, and it is the only local move the algorithm uses that still lacks a general proof: Lemma 4.1 covers the generic split and Lemma 4.2 the order -3 case. Second, supply a closed-form family of base meshes with $\text{im } \rho = 2\mathbb{Z}_4$; we have explicit ones only for $2 \leq g \leq 8$, found by search rather than by formula, whereas Lemma 4.3 gives $d = 1$ and $d = 4$ in closed form for every genus. Third, and hardest: does step 2 of Algorithm 4.5 ever get stuck? Equivalently, does a cone of valence w on a mesh of genus ≥ 2 always carry a loop of the gap the next split wants, after remeshing if necessary? An affirmative answer would remove the dependence of Theorem 3.10 on (S7), reproving the whole classification constructively and with explicit minimal witnesses.
- (3) *Real strata.* Deciding the feature-curve case (§6.1) requires non-emptiness of strata of k -differentials invariant under a prescribed anti-holomorphic involution. All the evidence we have points at these being non-empty whenever Gauss–Bonnet and the parity constraint of Lemma 6.1 allow, which would make feature-aligned existence unconditional.
- (4) *Fixed connectivity.* Everything here allows refinement (Lemma 2.4). Existence at a fixed mesh connectivity is a different and probably much harder question.
- (5) *Quantization.* Square-tiled surfaces are exactly the seamless parametrizations that already are integer-grid maps, so the gap between Theorem 3.6 and “realizable by a mesh with at most N squares” is the quantization gap.

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